Ribbon R-Trees and Holomorphic Dynamics on the Unit Disk

Ribbon $\mathbb{R}$-trees and holomorphic dynamics on the unit disk

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28 November, 2007

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1 Introduction

Let $\Delta \subset \mathbb{C}$ denote the unit disk, viewed as a model for the hyperbolic plane. Under rescaling, Δ takes on the appearance of a tree, with an additional *ribbon structure* coming from the cyclic ordering of its ends.

In this paper we show branched coverings of ribbon trees naturally compactify the space of proper holomorphic maps $f : (\Delta, 0) \to (\Delta, 0)$, and use the structure of these ribbon trees to describe the limiting moduli of f.

Ribbon $\mathbb{R}$-trees. We begin with some definitions. A metric space T is an $\mathbb{R}$-*tree* if any two points $x, y \in T$ are joined by a unique arc $[x, y] \subset T$ which is isometric to an interval in $\mathbb{R}$. It is *simplicial* if there is a discrete set of

*Research supported in part by the NSF.

vertices $V \subset T$ such that $T - V$ is a disjoint union of arcs. (Like an idealized conifer, a simplicial tree can have infinitely many edges attached to a single vertex.)

A *ribbon structure* on a finite simplicial tree S is the choice of a planar embedding $S \hookrightarrow \mathbb{R}^2$, up to isotopy. The ribbon structure can be specified by a cyclic ordering of the edges incident to each vertex. When T is equipped with compatible ribbon structures on all its finite subtrees, it becomes a *ribbon* $\mathbb{R}$*-tree*.

A map $f : T_1 \to T_2$ between ribbon $\mathbb{R}$-trees is a *branched covering* if f is a locally isometric covering map outside a finite set of critical points $C(f) \subset T_1$, where it is modeled on the map $(r, \theta) \mapsto (r, n\theta)$ in $\mathbb{R}^2$. The global degree of f is always finite.

Dynamics of branched coverings. Our first results (§4) describe the dynamics and translation lengths for branched coverings with a distinguished fixed point p.

Theorem 1.1 *Let $f : (T, p) \to (T, p)$ be a minimal branched covering of a ribbon $\mathbb{R}$-tree with* $\deg(f) \geq 2$*. Then T is simplicial, the orbits of f are discrete, and $T/\langle f \rangle$ is a finite tree with at most* $3|C(f)| - 1$ *edges.*

(Here *minimal* means (T, p) has no proper, totally invariant subtree (T', p).)

Let $\epsilon(T)$ denote the cyclically ordered space of ends of T. The *length function* $Lf : \epsilon(T) \to [0, \infty)$ measures the rate at which f moves points of T towards p; it is defined by

$$Lf(\alpha) = \lim_{x \to \alpha} d(x, p) - d(f(x), p).$$

In §7 we show:

Theorem 1.2 *The translation length Lf is a piecewise-constant function on the ends of T. Its finitely many values lie in the subgroup of $\mathbb{R}$ generated by the edge lengths of $T/\langle f \rangle$.*

Dynamics on the unit disk. Now let $f : \Delta \to \Delta$ be a proper holomorphic map on the unit disk of degree $d \geq 2$. Such a map has $(d-1)$ critical points $C(f) \subset \Delta$.

The action of the iterates of f on Δ is analogous to the action of a finitely-generated Fuchsian group $\Gamma \subset \mathrm{Aut}(\Delta)$. We will assume that f has a fixed point in the disk; this is analogous to the condition that the quotient

Riemann surface Δ/Γ is compact. Then up to conjugacy, f is a Blaschke product of the form

$$f(z) = z \prod_{1}^{d-1} \left(\frac{z - a_i}{1 - \overline{a}_i z} \right), \quad a_i \in \Delta. \tag{1.1}$$

Let $\mathcal{B}_d \cong \Delta^{(d-1)}$ denote the space of all such Blaschke products.

Length spectra. Any $f \in \mathcal{B}_d$ extends continuously to a measure-preserving covering map on the circle. There is a natural *marking homeomorphism* $\phi : S^1 \to S^1$, which conjugates f to the model mapping $p_d(z) = z^d$, and is normalized so $\phi(z) = z$ when $f = p_d$. We define the *length* on f of a periodic cycle C for p_d by

$$L(C, f) = \log |(f^q)'(z)|,$$

where $q = |C|$ and $\phi(z) \in C$.

The values $L(C, f)$, labeled by C, form the *marked length spectrum* of f; they determine f uniquely [SS]. (The analogous result for negatively curved surfaces is proved in [Ot1].)

A continuous version of the marked length spectrum is provided by the *length function*

$$Lf(z) = \log |f'(\phi^{-1}(z))|, \quad z \in S^1.$$

Note that

$$L(C, f) = \sum_{z \in C} Lf(z) \tag{1.2}$$

for each cycle C. It is often useful to normalize Lf by its maximum

$$M(f) = \sup_{z \in S^1} Lf(z),$$

which tends to infinity if f diverges in $\mathcal{B}_d$.

Geometric limits. By the Schwarz lemma, f contracts the hyperbolic metric on Δ; but far from its critical points, f is nearly an isometry. This observation furnishes a connection with trees, made precise in §11:

Theorem 1.3 *Any divergent sequence $f_n \in \mathcal{B}_d$ has a subsequence which converges geometrically to a degree d minimal branched covering $f : (T, p) \to (T, p)$ of a ribbon $\mathbb{R}$-tree.*

The tree T is constructed as the Gromov-Hausdorff limit of a sequence of hyperbolic polygons in Δ, namely the convex hulls of finite sets of the form

$$\bigcup_{i=-k}^{k} f_n^i(C(f_n) \cup \{0\}),$$

where $C(f_n)$ is the set of critical points of f_n. This construction is in the same spirit as case of the surface groups, treated in [Be1] and [Pau1].

In §12 we show the length function Lf for the limiting tree controls the limiting length spectra of f_n. Theorem 1.2 then allows one to deduce:

Theorem 1.4 *Suppose f_n diverges in $\mathcal{B}_d$. Then there is a subsequence such that*

$$L(z) = \lim_{n\to\infty} \frac{Lf_n(z)}{M(f_n)} \in [0,1]$$

exists for all $z \in S^1$. The limit $L(z)$ is a piecewise constant function with at most $(d-1)$ laps.

(The lap condition means $L(z)$ rises and falls at most $(d-1)$ times as z traverses S^1.)

Corollary 1.5 *The limiting cycle lengths*

$$L(C) = \lim_{n\to\infty} L(C, f_n)/M(f_n)$$

span a finite-dimensional vector space over $\mathbb{Q}$.

Algebraic limits. The space of Blaschke products of degree d also has a natural *algebraic compactification* $\overline{\mathcal{B}}_d \cong \overline{\Delta}^{(d-1)}$, whose boundary points (F, S) are pairs consisting of a map F on the disk and divisor of *sources* S on the circle.

The pair (F, S) has an associated *marking relation* $\Phi : S^1 \to S^1$, which blows the sources up to intervals and relates the dynamics of (F, S) to the dynamics of $p_d(z) = z^d$ [Mc3]. Under this marking, points that are about to escape from the influence of a given source $s \in \operatorname{supp} S$ are labeled by the interval

$$\Phi^*(s) = \{z \in \Phi(s) \, : \, z^d \notin \Phi(F(s))\} \subset S^1.$$

We say $f_n \to (F, S)$ *strongly* if its algebraic limit is (F, S), and its geometric limit is the tree obtained by *suspending* (F, S) (§5). Radial approach to $\partial\mathcal{B}_d$ guarantees strong convergence (§14).

In §13 we show that the length spectrum for a strong limit can be described directly in terms of the escape intervals for (F, S).

Theorem 1.6 *If $f_n \to (F,S) \in \partial\mathcal{B}_d$ strongly, then for almost all $z \in S^1$ we have*

$$L(z) = \lim \frac{Lf_n(z)}{M(f_n)} = \begin{cases} 1 & \text{if } z \in \Phi^*(s) \text{ for some source } s \in \operatorname{supp} S; \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the limiting length spectrum satisfies

$$L(C) = \lim \frac{L(C,f_n)}{M(f_n)} = \sum_{\operatorname{supp} S} |C \cap \Phi^*(s)| \in \mathbb{Z} \tag{1.3}$$

for almost all cycles C.

Here 'almost all' means finitely many exceptional z and C are excluded. When $|\operatorname{supp} S| = (d-1)$, equation (1.3) holds for all C.

Quadratic trees. We now turn to a more detailed discussion of the case $d = 2$. Here the space of branched coverings admits a particularly simple description.

Theorem 1.7 *The normalized, minimal quadratic branched coverings $f : (T,p) \to (T,p)$ are parameterized by the circle $\mathbb{R}/\mathbb{Z}$ with its rational points blown up to intervals.*

Indeed, in the quadratic case f has a unique critical point $c \in T$, and $c \neq p$ by minimality. We say f is *normalized* if $d(c,p) = 1$. Since f is a homeomorphism near p, it has a well-defined *rotation number* $\rho(f) \in S^1 = \mathbb{R}/\mathbb{Z}$. When $\rho(f) = a/b$ is rational, there is an additional invariant $\delta(f) \in [-1,1]$ which measures the failure of $f^b(c)$ to land on c. In §8 we show these two invariants determine f up to isomorphism.

Julia sets. Next we relate these quadratic trees, and the length spectra of maps in $\mathcal{B}_2$, to the Julia sets of quadratic polynomials.

For any rational $a/b \neq 0 \in S^1$, there is a unique parameter $c(a/b)$ in the a/b limb of the Mandelbrot set such that $P(z) = z^2 + c(a/b)$ satisfies $P^b(0) = 0$. Let

$$K(P) = \{z \in \mathbb{C} \,:\, \sup |P^n(z)| < \infty\}$$

denote its filled Julia set, and let $\Omega(P)$ be the interior of $K(P)$. The map P has a distinguished fixed point α which separates $K(P)$ into b components.

There are four external rays landing at α and $-\alpha$ adjacent to $z = 0$ in $K(P)$. The angles of these rays form the endpoints of 3 consecutive intervals

$$K_-(a/b), K(a/b), K_+(a/b) \subset S^1,$$

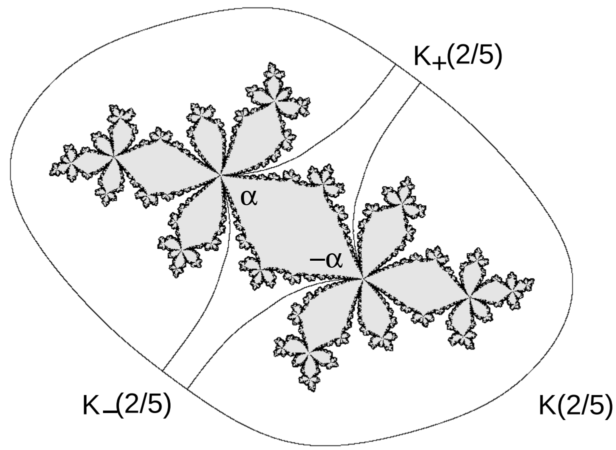


Figure 1. The filled Julia set for $P(z) = z^2 + c(2/5)$.

with $0 \in K(a/b)$. See Figure 1 for the case $a/b = 2/5$, where $K(a/b) = [41/62, 9/62]$.

The *full Hubbard tree* $f : (T, p) \to (T, p)$ records the combinatorial topology of $P|K(P)$. The tree T has a vertex for each component U of $\Omega(P)$, and an edge from U to U' when their closures meet. The action of P gives a simplicial branched covering map $f : T \to T$, and α labels a distinguished edge midpoint $p \in T$ fixed by f.

Let us say $u_n \in \Delta$ converges to $u \in S^1$ *no worse than quadratically* if for every $\epsilon > 0$, we have

$$|u - u_n|^{2+\epsilon} = O_\epsilon(1 - |u_n|).$$

This conditions holds, for example, if $u_n \to u$ radially or along a horocycle.

With these notions in hand, the behavior of the length spectrum of a diverging sequence $f_n \in \mathcal{B}_2$ can be described as follows.

Theorem 1.8 *Suppose $f_n'(0)$ converges, no worse than quadratically, to a root of unity $u = \exp(2\pi i a/b)$. Then f_n converges geometrically to the full Hubbard tree for $P(z) = z^2 + c(a/b)$, and its length spectrum satisfies*

$$L(C) = \lim \frac{L(C, f_n)}{M(f_n)} = |C \cap K(a/b)|$$

for all cycles C.

This theorem is a consolidation of results in §8, §9 and §14. The hypothesis insures that f_n converges strongly to its algebraic limit, whose unique source $s = -u$ satisfies

$$\begin{aligned} \Phi(s) &= K_-(a/b) \cup K(a/b) \cup K_+(a/b) \quad \text{and} \\ \Phi^*(s) &= K(a/b). \end{aligned}$$

Thus the formula given for $L(C)$ is also a special case of Theorem 1.6.

Irrational rotations. The definition of $K(a/b)$ extends by continuity to irrational numbers $t \in S^1$. In this case strong convergence is automatic, and we have:

Theorem 1.9 *Suppose $f_n'(0) \to u = \exp(2\pi i t)$ and t is irrational. Then $L(C) = |C \cap K(t)|$ for all cycles C.*

However the interval $K(t)$ does *not* vary continuously at rational points $t = a/b$. In fact, we have

$$\lim_{t \to a/b\pm} K(t) = K(a/b) \cup K_\pm(a/b).$$

The quadratic trees with rotation number $\rho(f) = a/b$ and $\delta(f) \in [-1, 1]$ serve exactly to interpolate between these two limits, and the results of §8 and §14 imply:

Theorem 1.10 *For any rational $a/b \in S^1$, $\delta \in [0, 1]$ and $\epsilon = \pm 1$, there is a sequence $f_n \in \mathcal{B}_2$ satisfying*

$$L(C) = |C \cap K(a/b)| + \delta |C \cap K_\epsilon(a/b)|$$

for all C except the unique cycle with rotation number a/b.

Corollary 1.11 *Every quadratic tree arises as a geometric limit of holomorphic maps.*

Comparison with surface groups. Let $\mathcal{T}_g$ denote the Teichmüller space of hyperbolic metrics on a surface S of genus $g \geq 2$. We conclude by comparing the results above with the well-known compactification of $\mathcal{T}_g$ by measured laminations, which are dual to $\mathbb{R}$-trees.

1. *Simpliciality.* By Theorem 1.1, whenever (T, p) supports a minimal branched covering f, the tree T is simplicial. This behavior is in sharp contrast to the case of surface groups: the $\mathbb{R}$-tree T coming from a measured lamination is almost never simplicial, and typically, every orbit of $\pi_1(S)$ in T is dense.

2. *Finiteness.* Nevertheless the analogue of Corollary 1.5 holds for laminations: the intersection numbers

$$\{i(C, \lambda) \,:\, C \text{ is a closed curve on } S\}$$

span a finite-dimensional vector space over $\mathbb{Q}$. (This is easily established using train tracks.)

3. *Markings.* The action of a surface group on an $\mathbb{R}$-tree immediately yields a translation length for every closed curve C on S. The translation lengths for a branched covering $f : (T, p) \to (T, p)$ are more difficult to extract; they are labeled by periodic cycles C for $z^d|S^1$, which may or may not correspond to periodic ends of T.

4. *Currents.* The space of measured laminations can be constructed by embedding $\mathcal{T}_g$ into the projective space of invariant measures for the geodesic flow on $T_1(S)$, and taking the closure [Bon2]. Similarly, for Blaschke products the map

$$f \mapsto \nu_f = \phi_*(|dz|/2\pi)$$

embeds $\mathcal{B}_d$ into the space $M_d(S^1)$ of invariant measures for $z^d|S^1$; passing to the closure, we obtain a compactification whose boundary is a sphere [Mc3].

This boundary is different, however, from the one provided by branched coverings of trees. For example, when $d = 2$ the boundary of $\mathcal{B}_2$ in $M_2(S^1)$ is parameterized by S^1, while the space of quadratic trees is parameterized by S^1 with its rational points blown up (Theorem 1.7).

5. *Intersection number.* The geometric intersection number on surfaces unifies the discussion of metrics and laminations. For example, a closed curve C and a hyperbolic surface X both determine currents, and their intersection number $i(C, X)$ is simply the length of C on X.

In the setting of Blaschke products, we have the integration pairing

$$I : M_d(S^1) \times L^\infty_b(S^1) \to \mathbb{R}$$

between z^d-invariant measures and bounded Borel functions. Regarding a periodic cycle as a sum of δ-masses, equation (1.2) expresses the length of a cycle as the intersection

$$L(C, f) = I(C, Lf).$$

Furthermore the thermodynamic formalism allows one to reconstruct the invariant measure ν_f from its log-derivative Lf. However the symmetry of the intersection number seems to be missing, and (as just remarked) the embeddings of $\mathcal{B}_d$ into $M_d(S^1)$ and $L_b^\infty(S^1)$ lead to qualitatively different compactifications.

Notes and references. For more on valuations, $\mathbb{R}$-trees, measured foliations, degenerations of hyperbolic manifolds and the boundary of Teichmüller space, see e.g. [MS1], [MS2], [Be1], [Be2], [Pau1], [Pau2] and [Ot2]. Finite *ribbon graphs* play an important role in the topology of moduli spaces; see for example [Pen] and [Ko]. Other applications of trees in complex dynamics appear in [Shi], [Em], and [DeM].

A divergent sequence $f_n \in \mathcal{B}_d$ can also be studied via a direct analysis of its dynamics on the circle, which takes places at widely disparate scales and exhibits highly localized expansion. Ribbon $\mathbb{R}$-trees provide both a calculus of infinitesimals and a combinatorial framework to capture this structure. For an explicit connection between $\mathbb{R}$-trees and nonstandard analysis, see [Ch].

The fact that the trees compactifying $\mathcal{B}_d$ are simplicial suggests they are related to discrete valuations [Ki2], algebraic boundaries for the moduli spaces of rational maps [Sil], [Mil1], and their blowups [D1], [D2]. Tropical geometry may also provide a useful perspective on these compactifications (see [FG] and the proof of Theorem 12.1).

This paper is a sequel to [Mc2], [Mc3] and [Mc4], which pursue the connection between Blaschke products and other aspects of Teichmüller theory such as the Weil-Petersson metric, geodesic currents and lengths of simple closed curves.

The theory of ribbon $\mathbb{R}$-trees was developed independently in [Wo] for applications to representations of surface groups.

I would like to thank M. Bestvina and F. Paulin for useful correspondence.

Notation. The usual notations $A = O(B)$ and $A \asymp B$ mean $A < CB$ and $B/C < A < CB$, for an implicit constant C. The notation $A = O_s(B)$ means C is allowed to depend on s.

2 Cyclically ordered sets

This section records definitions and basic properties of cyclically ordered sets and their endomorphisms.

Translations. Let $\widetilde{A}$ be a totally ordered set. A *translation* is an order-preserving bijection $T : \widetilde{A} \to \widetilde{A}$ such that for each $x_0, x \in \widetilde{A}$, there is a unique $n \in \mathbb{Z}$ such that

$$T^n(x_0) \leq x < T^{n+1}(x_0). \tag{2.1}$$

Once the basepoint x_0 is fixed, we write $n = \lfloor x \rfloor$. Changing the basepoint only changes $\lfloor x \rfloor$ by a uniformly bounded amount.

Cyclic orderings. A *cyclic ordering* on a set A is a collection of subsets $(a, b) \subset A$, defined for each pair of distinct points $a, b \in A$, such that:

1. $\{(a, b), (b, a)\}$ is a partition of $A - \{a, b\}$,
2. $x \in (a, b) \implies b \in (x, a)$, and
3. the relation on (a, b) defined by $x < y$ if $(x, y) \subset (a, b)$ defines a total ordering on (a, b).

We regard (a, b) as the positively oriented open interval from a to b. These open intervals generate a natural topology on A. (When $|A| \leq 2$ we give A the discrete topology.)

Universal covers. The quotient $A = \widetilde{A}/\langle T \rangle$ of a totally ordered set by a translation has a natural cyclic ordering: we define (a, b) to be the projection of the open interval $(\widetilde{a}, \widetilde{b}) \subset \widetilde{A}$, for any pair of lifts of a, b such that $\widetilde{a} \leq \widetilde{b} < T(\widetilde{a})$.

Conversely, every cyclically ordered set has a totally ordered *universal cover* $\widetilde{A}$, equipped with a translation T such that $A \cong \widetilde{A}/\langle T \rangle$.

The basic example of a cyclically ordered set is $S^1 = \mathbb{R}/\mathbb{Z}$. Any cyclically ordered set with a countable dense subset can be embedded in S^1 so its cyclic ordering is preserved.

Maps. Let $f : A_1 \to A_2$ be a map between cyclically ordered sets with universal covers $(\widetilde{A}_1, T_1)$ and $(\widetilde{A}_2, T_2)$. We say f is a *covering map of degree* d if it lifts to an order-preserving bijection $\widetilde{f} : \widetilde{A}_1 \to \widetilde{A}_2$ satisfying

$$\widetilde{f}(T_1(x)) = T_2^d(\widetilde{f}(x)). \tag{2.2}$$

An *isomorphism* is a covering map of degree one.

We say f is *monotone increasing* if it is covered by a map $\widetilde{f} : \widetilde{A}_1 \to \widetilde{A}_2$ such that

$$x \leq y \implies \widetilde{f}(x) \leq \widetilde{f}(y)$$

and $\widetilde{f}(T_1(x)) = T_2(\widetilde{f}(x))$.

Markings. Let $f : A \to A$ be a covering map of degree $d > 1$, and assume $A \neq \emptyset$. Define $p_d : S^1 \to S^1$ by $p_d(t) = dt \bmod 1$.

A *semiconjugacy* between f and p_d is a monotone increasing map $\phi : A \to S^1$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \phi\big\downarrow & & \phi\big\downarrow \\ S^1 & \xrightarrow{p_d} & S^1 \end{array}$$

commutes. We say ϕ is *strong* semiconjugacy if $p_d|\phi(A)$ is d-to-one; equivalently, if $\phi(f^{-1}(x)) = p_d^{-1}(\phi(x))$ for all $x \in A$. (This condition implies $\phi(A)$ is dense in S^1; a semiconjugacy which is not strong might send all of A to a fixed point of p_d.)

Fix a basepoint $x_0 \in \widetilde{A}$, and define $\lfloor x \rfloor$ via (2.1). Choose a lift $\widetilde{f}$ of f satisfying (2.2), and define $\widetilde{\phi} : \widetilde{A} \to \mathbb{R}$ by

$$\widetilde{\phi}(x) = \lim_{n\to\infty} \frac{\lfloor \widetilde{f}^n(x) \rfloor}{d^n}. \tag{2.3}$$

It is easily verified that $\widetilde{\phi}(T(x)) = \widetilde{\phi}(x) + 1$, and that $\widetilde{\phi}$ descends to a strong semiconjugacy between f and p_d. Conversely, every strong semiconjugacy arises via this construction. We refer to the choice of one such ϕ as a *marking* of $f : A \to A$. (Cf. [Sh], [Mc4] for the case $A = S^1$.)

The map ϕ is independent of the choice of basepoint x_0. If we replace $\widetilde{f}$ by $T \circ \widetilde{f}$, then $\phi(x)$ is replaced by $\phi(x) + 1/(d-1)$; thus we have:

Proposition 2.1 *Any degree d covering map $f : A \to A$ admits a marking, and any two markings are related by the action of the group $\mathbb{Z}/(d-1)$ of automorphisms of p_d.*

Rotation numbers. Now let $f : A \to A$ be a monotone increasing map, and $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ a lift commuting with the action of T. As in the case of a circle homeomorphism (cf. [CFS, Ch. 3.3]), we define the *rotation number* of f by

$$\rho(f) = \lim_{n\to\infty} \frac{\lfloor \widetilde{f}^n(x) \rfloor}{n} \bmod 1 \tag{2.4}$$

for any $x \in \widetilde{A}$.

It is easily verified that the value of $\rho(f) \in S^1$ is independent of the choice of x and the basepoint x_0. If f has a periodic point, then $\rho(f)$ is rational, but not conversely. For example, the map $f(1/n) = 1/(n+1)$ on $A = \{1/n \bmod 1 \ : \ n > 0\} \subset S^1$, with the induced cyclic ordering, has $\rho(f) = 0$.

3 $\mathbb{R}$-trees and ribbon trees

In this section we define branched coverings between ribbon $\mathbb{R}$-trees and discuss their basic properties.

$\mathbb{R}$-trees. An $\mathbb{R}$-*tree* is a nonempty metric space (T, d) such that any two points $x, y \in T$ are connected by a unique topological arc $[x, y] \subset T$, and every arc in T is isometric to an interval in $\mathbb{R}$.

We say x is an *endpoint* of T if $T - \{x\}$ is connected; otherwise x is an *interior* point. If $T - \{x\}$ has three or more components, we say x is a *branch point*. The set of branch points will be denoted $B(T)$. We write $[x, y)$ and (x, y) for $[x, y]$ with one or both of its endpoints removed.

Convexity. A subset $S \subset T$ is *convex* if $x, y \in S \implies [x, y] \subset S$. The smallest convex set containing $E \subset T$ is its *convex hull*, denoted $\mathrm{hull}(E)$. We say E *spans* T if $T = \mathrm{hull}(E)$.

A subset $S \subset T$ is convex $\iff$ S is connected $\iff$ S is a subtree of T. Thus any continuous map $f : T_1 \to T_2$ preserves convex sets.

Simplicial trees. An $\mathbb{R}$-tree T is *simplicial* if there is a discrete set V spanning T such that the components of $T - V$ are arcs. In this case we say V is a *vertex set* for T, and the components of $T - V$ are *edges*. Every $x \in V$ has a neighborhood which is homeomorphic to a cone over a discrete set. The number of edges incident to a given vertex is its *degree*, which may be infinite; thus a simplicial tree need not be locally compact.

A continuous map $f : T_1 \to T_2$ is *simplicial* if there are vertex sets $V_i \subset T_i$ such that

$$f : (T_1 - V_1) \to (T_2 - V_2)$$

is a locally isometric covering map. Equivalently, T_1 and T_2 can be given simplicial structures such that f sends edges to edges, preserving their lengths.

Finite trees. A *finite tree* is a compact simplicial tree. The convex hull of any finite set $E \subset T$ is a finite tree, so any $\mathbb{R}$-tree is a direct limit of finite trees.

A *tripod* is a finite tree with exactly three endpoints.

Ribbon trees. A finite *ribbon tree* is a finite tree with a specified cyclic ordering of the edges around each branch point. Equivalently, T is equipped with a planar embedding up to isotopy. The ribbon structure determines a natural cyclic ordering on the endpoints of T.

Every subtree of a finite ribbon tree has an induced ribbon structure. A *ribbon* $\mathbb{R}$-*tree* is an $\mathbb{R}$-tree T with compatible ribbon structures on all of its finite subtrees.

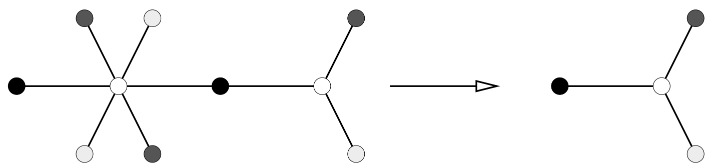


Figure 2. A degree 3 branched covering map between ribbon trees.

Branched coverings. Let $f : T_1 \to T_2$ be a simplicial map between finite ribbon trees relative to vertex sets $V_i \subset T_i$. The ribbon structure determines a cyclic ordering for the edges around each vertex $x \in V_i$ (note that only one ordering is possible when x is not a branch point.)

We say f is a *branched covering* if for each $x \in V_1$, f induces a covering map of positive degree from the cyclically ordered set of edges incident to x to those incident to $f(x)$. See Figure 2.

Equivalently, $f : T_1 \to T_2$ is a branched covering if there are planar embeddings of T_1 and T_2, compatible with their ribbon structures, such that f extends to a branched covering of $\mathbb{R}^2$ to itself. (The extension can even be taken to be a complex polynomial on $\mathbb{R}^2 \cong \mathbb{C}$ by [Thom].)

More generally, we say a map $f : T_1 \to T_2$ between ribbon $\mathbb{R}$-trees is a branched covering if for any finite tree $S \subset T_2$, the preimage $f^{-1}(S) = S_1 \cup \cdots \cup S_n$ is a finite union of finite trees, and $f : S_i \to S$ is a branched covering for each component of the preimage.

Critical points. Let $f : T_1 \to T_2$ be a branched covering between ribbon $\mathbb{R}$-trees. The *critical points* $C(f) \subset T_1$ are the points where f fails to be a local homeomorphism; their images are the *critical values* of f. The *degree* of f, denoted $\deg(f)$, is the maximum cardinality of its fibers. A critical point $x \in C(f)$ has *multiplicity* e if f is locally $(e+1)$-to-1 at x.

Basic properties. Any branched covering $f : T_1 \to T_2$ has the following properties.

1. The map f is open, its degree is finite, and f has $\deg(f) - 1$ critical points counted with multiplicity.

2. If $\deg(f) = 1$, then f is a global isometry preserving the ribbon structure.

3. Let $E_2 \subset T_2$ be the finite set of critical values of f, and let $E_1 = f^{-1}(E_2)$; then
$$f : (T_1 - E_1) \to (T_2 - E_2) \tag{3.1}$$

is a locally isometric covering map.

4. We have $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in T$.

5. If $[x, y]$ contains no critical points, then $[x, y]$ maps isometrically to $[f(x), f(y)]$; in particular, $d(x, y) = d(f(x), f(y))$.

6. More generally, any subtree S disjoint from $C(f)$ maps isometrically to its image $f(S)$.

7. If $S \subset T$ is a finite tree, then so is $f(S)$. Every endpoint of $f(S)$ is either a critical value of f or the image of an endpoint of S.

These facts are readily verified when T_1 and T_2 are finite; they follow for general trees by passing to the limit.

For later use (in Proposition 4.23) we also record:

Proposition 3.1 *Let $f : (T, p) \to (T, p)$ be a ribbon automorphism fixing p. Then all periodic points $q \in T - \{p\}$ have the same period.*

Proof. Suppose $q, q' \in T - \{p\}$ have periods n and n' respectively. Restricting our attention to the span of p and the forward orbits of q and q', we can assume T is a finite tree, and $f^n : T \to T$ fixes the arc $[p, q]$. Now whenever f^n fixes an edge e of T with vertex v, it also fixes the other edges adjacent to e, since their cyclic order is preserved. Consequently f^n fixes the entire tree, including q'; therefore $n'|n$. Reversing the roles of q and q', we find $n|n'$ and hence $n = n'$, so all periodic points in $T - \{p\}$ have the same order. ∎

Note that a ribbon automorphism of an infinite tree can have a mix of periodic and non-periodic points; see Figure 3 and Proposition 4.5.

The ends of a tree. A *ray* α in an $\mathbb{R}$-tree T is a subtree isometric to $[0, \infty) \subset \mathbb{R}$. Two rays are *equivalent* if $\alpha_1 \cap \alpha_2$ is again a ray. The collection $\epsilon(T)$ of all such equivalence classes forms the set of *ends* of T. We let α denote both a ray and the end it represents.

The ends of a ribbon tree have a natural cyclic ordering, inherited from the ordering on the endpoints of finite subtrees. We say a sequence $x_i \in T$ converges to the end represented by a ray $\alpha \subset T$ if $d(x_1, x_i) \to \infty$, and $x_i \in \alpha$ for all $i \gg 0$.

Now let $f : T_1 \to T_2$ be a degree d branched covering between ribbon $\mathbb{R}$-trees. Note that any end in $\epsilon(T_1)$ is represented by a ray α disjoint from $C(f)$, whose image

$$f_*\alpha = f(\alpha) \in \epsilon(T_2)$$

represents a well-defined end of T_2. The covering property expressed in equation (3.1) readily implies:

Proposition 3.2 *A branched covering $f : T_1 \to T_2$ determines a covering map*

$$f_* : \epsilon(T_1) \to \epsilon(T_2),$$

with the same degree as f.

4 Dynamics on trees

Let (T, p) be a ribbon $\mathbb{R}$-tree with a distinguished point $p \in T$. In this section we prove a structure theorem for the dynamics of a branched covering

$$f : (T, p) \to (T, p)$$

fixing p, with $\deg(f) \geq 2$.

Definitions. We say f is *minimal* if there is no proper subtree of T containing p and invariant under f^{-1}. Points $x, y \in T$ belong to the same *grand orbit* if $f^i(x) = f^j(y)$ for some $i, j > 0$. We let $\overline{x} \subset T$ denote the grand orbit of x, and $\overline{E}$ the union of the grand orbits meeting $E \subset T$. A grand orbit $\overline{x}$ is *periodic* if $f^i(x) = f^j(x)$ for some $i > j > 0$.

Let $T/\langle f\rangle = \{\overline{x} : x \in T\}$ be the space of grand orbits, with the distance function

$$d(\overline{x}, \overline{y}) = \inf\{d(x, y) : x \in \overline{x}, y \in \overline{y}\}, \tag{4.1}$$

and let $\pi : T \to T/\langle f\rangle$ be the natural projection map.

The structure theorem essentially states that a minimal branched covering has finite combinatorial complexity.

Theorem 4.1 *Let $f : (T, p) \to (T, p)$ be a minimal branched covering of a ribbon $\mathbb{R}$-tree, and let*

$$V = \overline{B}(T) \cup \overline{C}(T) \cup \overline{p}$$

be the union of the grand orbits of the branch points of T, the critical points of f and the basepoint p. Then:

1. *T is a simplicial tree with respect to the vertex set V;*
2. *$f : T \to T$ is simplicial with respect to V;*
3. *V consists of at most $3|C(f)|$ grand orbits, of which at most $2|C(f)|$ are periodic;*

4. $T/\langle f\rangle$ *is a finite tree, and*

5. $\pi : T \to T/\langle f\rangle$ *is an open, simplicial map with respect to the vertex sets* V *and* $V/\langle f\rangle$.

Corollary 4.2 *The edges forming* $T - V$ *fall into at most* $3|C(f)|-1$ *grand orbits, of which at most* $2|C(f)|-1$ *are periodic.*

Corollary 4.3 T *is a complete metric space.*

Minimality. We begin the proof of Theorem 4.1 with a characterization of minimality.

Proposition 4.4 *A branched covering* $f : (T,p) \to (T,p)$ *with* $\deg(f) \geq 2$ *is minimal if and only if*

$$T = \bigcup_{c\in C(f)} \overline{[c,p]}.$$

Proof. Let $S = \bigcup_{C(f)}[c,p]$. Since $\deg(f) \geq 2$, S is nonempty. Note that $S^+ = \bigcup_{i\geq 0} f^i(S)$ is connected and contains all the critical values of f. Thus $\overline{S} = \bigcup_j f^{-j}(S^+)$ is also connected (as can be verified by induction on j). Since $\overline{S}$ is a totally invariant subtree, if f is minimal, then $\overline{S} = T$.

Conversely, suppose $\overline{S} = T$ and R is a totally invariant subtree containing p. We may assume R contains a nontrivial arc (otherwise $C(f) = \{p\}$ and hence $\overline{S} = R = T$). Thus R contains a point with d preimages, and hence $\deg(f|R) = d$, which implies $C(f) \subset R$. But R is convex, so we have $\bigcup_{C(f)}[c,p] \subset R$, and hence $T = \overline{S} \subset R$ by total invariance of R. Consequently f is minimal. ∎

Ribbon property. It is useful to introduce the radius function $r(x) = d(x,p)$. The next result is the main step in the proof that uses the ribbon structure on T.

Proposition 4.5 *Let* $f : (T,p) \to (T,p)$ *is a branched covering, and suppose* $r(f^i(x)) = r(x)$ *for all* $i \geq 0$. *Then there is a unique* $y \in [x,p]$ *such that the arc* $[y,p]$ *is preperiodic, while the arcs* $f^i([x,y)), i = 0,1,2,3,\ldots$ *are disjoint.*

Proof. Let $x_i = f^i(x)$, $i \geq 0$. Since $d(x_i,p) = d(x,p)$ for all i, the map f sends $[x_i,p]$ isometrically to $[x_{i+1},p]$ for each i. In particular, we have

$$r(z) = r(f(z)) = r(f^2(z)) = \cdots \tag{4.2}$$

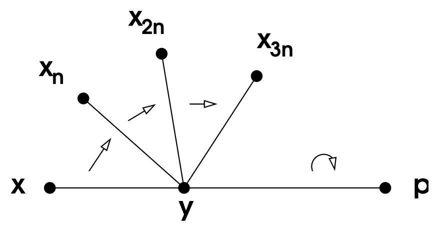


Figure 3. The fan formed by an edge with an infinite forward orbit.

for all $z \in [x,p]$. Thus distinct points in $[x,p]$ belong to distinct grand orbits; in particular, $[x,p] \cap \overline{C}(f)$ is finite.

Let Y be the union of p and the preperiodic points in $[x,p] \cap \overline{C}(f)$. Since Y is finite, there is point $y_0 \in Y$ closest to x. If there are no preperiodic points in $[x, y_0)$, then we can simply take $y = y_0$ to complete the proof.

Otherwise there is a preperiodic point $y_1 \in [x, y_0)$. Replacing x with a suitable forward iterate, we can assume y_1 is actually periodic; say $f^n(y_1) = y_1$. Since T is a tree, we have

$$[x,p] \cap [x_n, p] = [y, p]$$

for some y which is also fixed by f^n. Since $y_1 \in [y,p]$, y is not a critical point of f and $y \neq p$.

If $y = x$ the proof is again complete. Otherwise, the points (x, x_n, p) span a tripod $S \subset T$ with y in its interior (see Figure 3). Since $y \notin C(f)$, f^n preserves the cyclic ordering of the edges of T adjacent to y, and it fixes the edge $[y,p]$. But $[x,y]$ and $[x_n, y]$ meet only at y, so the edges $[x,y]$, $[x_n,y]$, $[x_{2n}, y]$, $[x_{3n}, y], \ldots$ must form an infinite fan based at y. This shows every point in $[x,y)$ has an infinite forward orbit, completing the proof in this case as well. ∎

Corollary 4.6 *Any recurrent point for f is periodic.*

Proof. Let $x_i = f^i(x)$ and suppose x is recurrent; that is, $\liminf d(x, x_i) = 0$. Then $r(x) = r(x_i)$ for all i. Let y be the preperiodic point in $[x,p]$ closest to x, as above. Then $[x, x_i]$ must pass through y for all i; hence $y = x$, which implies x is periodic. ∎

Corollary 4.7 *The periodic points of f form a closed subset of T.*

Proof. Since f does not increase distances, any limit of periodic points is recurrent, and hence periodic. ■

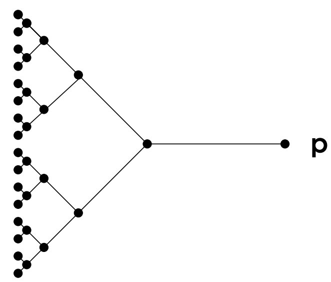


Figure 4. A bifurcating tree with ends $\mathbb{Z}_2$.

Example: The adding machine. The preceding results fail if we do not require that f preserves a ribbon structure on T. For example, the Cantor set of 2-adic integers $\mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^n$, together with a disjoint basepoint p, can be realized as the endpoints of a bifurcating tree T (Figure 4) in such a way that the map $f(x) = x + 1$ on $\mathbb{Z}_2$ extends to an automorphism of (T, p). Then every $x \in \mathbb{Z}_2 \subset T$ is recurrent under f, but none is periodic. Moreover every $x \in T - \mathbb{Z}_2$ is periodic, so the periodic points of f are not a closed subset of T.

Metric on the quotient. Next we prove that (4.1) defines a metric on $T/\langle f \rangle$. We begin with:

Proposition 4.8 *The forward orbit of any point $x \in T$ is discrete.*

Proof. Any accumulation point y of $\langle x_i \rangle = \langle f^i(x) \rangle$ is recurrent, and hence periodic. Let Y be the finite forward orbit of y, and let $2r$ be the minimum distance between any pair of distinct points in $Y \cup C(f)$. Then as soon as $d(x_i, Y) < r$, there are no critical points between x_i and its nearest point in Y, and hence $d(x_{i+1}, Y) = d(x_i, Y)$. In other words, $d(x_i, Y)$ is eventually constant. Since x_i accumulates on Y, this means $x_i = y$ for i large enough, and hence the forward orbit of x is finite. ■

Proposition 4.9 *The precritical points $C^-(f) = \bigcup_0^\infty f^{-i}(C(f))$ form a discrete subset of T.*

Proof. Suppose y is a limit point of $C^-(f)$. For each i, let x_i be one of the points in the finite set $\bigcup_{j=0}^{i} f^{-j}(C(f))$ that is closest to y, but not equal to y. Then $f^{n_i}(x_i) \in C(f)$ for some $n_i \le i$, and $x_i \to y$. Pass to a subsequence such that $f^{n_i}(x_i) = c \in C(f)$ is constant, and let $y_i = f^{n_i}(x_i)$. Then $y_i \to c$ but

$$d(c, y_i) = d(x_i, y) \neq 0,$$

since f^{n_i} has no critical points in $[x_i, y]$. This contradicts the discreteness of the forward orbit of y. ∎

Proposition 4.10 *Let $S \subset T$ be a finite subtree, and let $S_i = f^i(S)$. Then $f : S_i \to S_{i+1}$ is an isometry for all i sufficiently large.*

Proof. By the basic properties of branched coverings (§3), each S_i is a finite tree and $f : S_i \to S_{i+1}$ is simplicial. Let $E_i = S_i \cap C(f)^-$ denote the set of points in S_i that land on critical points of f under forward iteration. Since $C(f)^-$ is discrete and S_i is compact, E_i is finite. Note that the full preimage of E_{i+1} inside S_i lies inside E_i; in particular, $|E_{i+1}| \le |E_i|$.

Let e_i denote the sum over $x \in E_i$ of the number of edges of S_i incident to x. The edges counted by e_{i+1} are all images of edges contributing to e_i, so $e_{i+1} \le e_i$.

Choose n so $|E_i|$ and e_i are constant for all $i \ge n$. We claim $f : S_i \to S_{i+1}$ is an isometry for all $i \ge n$. To see this, consider any point $x \in E_i$ with $i \ge n$. Since $|E_i| = |E_{i+1}|$, no point in S_i other than x maps to $f(x)$. Since $e_i = e_{i+1}$, the edges adjacent to x in S_i map to distinct edges adjacent to $f(x)$. Thus f is a local homeomorphism near x. Since the critical points of $f|S_i$ are contained in E_i, $f|S_i$ is also a local homeomorphism at the other points of S_i. Since $S_{i+1} = f(S_i)$ is a tree, this implies $f|S_i$ is a global homeomorphism, and hence an isometry by the definition of a branched covering. ∎

Corollary 4.11 *For any $x \in T$, the sequence $r(f^i(x))$ is eventually constant.*

Proof. Apply the preceding result with $S = [x, p]$. ∎

By Proposition 4.5 this implies:

Corollary 4.12 *The forward orbit* $X = \{x, f(x), f^2(x), \ldots\}$ *of any* $x \in T$ *is uniformly discrete.*

(This means there is an $r > 0$ such that $d(x_1, x_2) > r$ for all pairs $x_1 \neq x_2$ in X.)

Proposition 4.13 *For any* $x, y \in T$*, we have* $d(\overline{x}, \overline{y}) = 0$ *iff* $\overline{x} = \overline{y}$*.*

Proof. Choose $r > 0$ such that distinct points in the forward orbit X of x are separated by at least distance r. Since f has only finitely many critical points, there is also an $s > 0$ such that $B(x, s) \cap C(f) \subset \{x\}$ for all $x \in X$.

Let $y_i = f^i(y)$; then $d(\overline{x}, \overline{y}) = \lim_{i\to\infty} d(X, y_i)$. Suppose this limit is zero. Since the points of X are well-separated, once i is large enough there is a unique point $x_i \in X$ with $d(x_i, y_i) < r/2$. But since f is distance-nonincreasing, we must have $x_{i+1} = f(x_i)$. Thus $d(x_i, y_i) \to 0$.

Choose n such that $d(x_i, y_i) < s$ for all $i \geq n$. Then $[x_i, y_i]$ has no critical points in its interior. Consequently $d(x_i, y_i) = d(x_{i+1}, y_{i+1})$ for all $i \geq n$. Thus $d(x_n, y_n) = 0$, and therefore $\overline{x} = \overline{y}$. ■

Corollary 4.14 *The quotient* $T/\langle f\rangle$ *is a metric space.*

Map to the quotient is a tree. We now analyze the geometry of $\pi : T \to T/\langle f\rangle$ to prove $T/\langle f\rangle$ is a finite tree. We begin with a statement about the fibers of π.

Proposition 4.15 *For any* $x \in T$*, the grand orbit* $\overline{x}$ *is uniformly discrete.*

Proof. Since $T/\langle f\rangle$ is a metric space and f has only finitely many critical points, we have $d(\overline{x}, \overline{C}(f) - \overline{x}) = s > 0$. Using Corollary 4.12, choose $r > 0$ so distinct points in the forward orbit X of x are separated by distance at least r. Suppose $y, z \in \overline{x}$ and $d(y, z) < \min(r, s)$. Choose $i > 0$ such that $f^i(y), f^i(z) \in X$. Since these points are closer than r, we have $f^i(y) = f^i(z)$; but since $d(y, z) < s$, f^i has no critical points in the interior of $[y, z]$, and thus $y = z$. ■

Proposition 4.16 *Let $S \subset T$ be a subtree containing p. If $\pi|S$ is injective, then $\pi|S$ is an isometry.*

Proof. Since $\pi|S$ is injective, so is $f^i|S$ for all $i \geq 1$, and hence $f^i|S$ is an isometry. In particular, $r(f^i(x)) = r(x)$ for all $x \in S$, and $d(x,z) = r(x) - r(z)$ if $z \in [x,p]$.

We wish to show that for every $x, y \in S$, we have $d(\overline{x}, \overline{y}) = d(x,y)$; equivalently, that for any $j, k \geq 0$ we have

$$d(x_j, y_k) \geq d(x,y). \tag{4.3}$$

Interchanging x and y if necessary, we can assume $j > k$; and replacing S with $f^k(S)$, we can assume $k = 0$.

Let $[x_i, p] \cap [y_0, p] = [z_i, p]$ for $i \geq 0$. Since $\pi|S$ is injective, the arcs $[x_0, z_0)$ and $[y_0, z_0)$ have distinct grand orbits and hence $r(z_i) \leq r(z_0)$. Since T is a tree, we have $z_i \in [x_i, y_0]$; therefore

$$\begin{aligned} d(x_i, y_0) &= d(x_i, z_i) + d(z_i, y_0) = r(x_i) + r(y_0) - 2r(z_i) \\ &\geq r(x_i) + r(y_0) - 2r(z_0) = r(x_0) + r(y_0) - 2r(z_0) = d(x_0, y_0) \end{aligned}$$

for all $i \geq 0$. In particular we have $d(x_j, y_0) = d(x_j, y_k) \geq d(x,y)$, as desired. ∎

Proposition 4.17 *If $f^i|[a,p]$ is an isometry for all i, then $\pi|[a,p]$ is an isometry.*

Proof. Given $x, y \in [a,p]$, let x_i and y_i denote $f^i(x)$ and $f^i(y)$. Then $r(x_i) = r(x)$ and $r(y_i) = r(y)$, and therefore

$$d(x,y) \geq d(\overline{x}, \overline{y}) = \inf_{i,j>0} d(x_i, y_j) \geq \inf |r(x_i) - r(y_j)| = |r(x) - r(y)| = d(x,y).$$

∎

Proposition 4.18 *Suppose $f^i|[a,p]$ and $f^i|[b,p]$ are isometries for all $i \geq 0$. Then there exists an $i, j \geq 0$ such $\pi|[f^i(a), p] \cup [f^j(b), p]$ is an isometry.*

Proof. Let $A \subset [a,p]$ and $B \subset [b,p]$ be the collections of points which are identified under projection to $T/\langle f \rangle$. Since π is continuous and $T/\langle f \rangle$ is a metric space, the sets A and B are closed.

Let $a_0 \in A$ be the point closest to a. Then $f^i(a_0) = f^j(b_0)$ for some $b_0 \in [b, p]$. Since $f^i|[a, p]$ and $f^j|[b, p]$ are isometries, this implies

$$f^i([a_0, p]) = f^j([b_0, p]) = f^i(A) = f^j(B).$$

Thus $\pi|[f^i(a), p] \cup [f^j(b), p]$ is injective, and hence an isometry. ∎

The next result is the only step in the proof of the structure theorem that uses minimality.

Proposition 4.19 *Suppose f is minimal. Then there exists a finite subtree $S \subset T$ such that π maps S isometrically onto $T/\langle f \rangle$.*

Proof. We may assume $\deg(f) \geq 2$. Let $\mathcal{S}$ denote the family of finite subtrees $S \subset T$, with ordered endpoints $p, e_1, \ldots, e_n$, such that:

1. $\pi(S) = T/\langle f \rangle$, and
2. $f^i|E_j$ is an isometry for all $i \geq 0$ and $j = 1, \ldots, n$,

where $E_j = [e_j, p]$.

The set $\mathcal{S}$ is nonempty; indeed, by Propositions 4.4 and 4.10, we have $f^i(\bigcup_{C(f)}[c, p]) \in \mathcal{S}$ for all $i \gg 0$.

Let $\mathcal{S}_0 \subset \mathcal{S}$ be the subfamily where number of endpoints n of S is minimized. Choose $S \in \mathcal{S}_0$ and $m \geq 1$ such that $\pi|(E_1 \cup E_2 \cup \cdots \cup E_m)$ is injective on the first m edges of S, and the value of m is maximized over all $S \in S_0$. If $m = n$, then we are done.

Otherwise, there is a point $a \in (E_1 \cup \cdots \cup E_m) - E_{m+1}$ and a $b \in E_{m+1}$ such that $\pi(a) = \pi(b)$. Choose a such that $r(a) = r(b) \leq r(e_{m+1})$ is large as possible (recall that the pairs of identified points form a closed set). Then $a \in E_i$ for some $i \leq m$.

By Proposition 4.18, there exists a $j, k > 0$ such that π is injective on $f^j(E_i) \cup f^k(E_{m+1})$. Let

$$S' = f^j(E_1 \cup \cdots E_m) \cup f^k(E_{m+1} \cup \cdots E_n),$$

with ordered endpoints

$$(e'_1, \ldots, e'_n) = (f^j(e_1), \ldots, f^j(e_m), f^k(e_{m+1}), \ldots, f^k(e_n)).$$

These endpoints are distinct since n was minimal. Thus $S' \in \mathcal{S}_0$. But now $\pi|(E'_1 \cup \cdots \cup E'_{m+1})$ is injective, contradicting the maximality of m. ∎

Corollary 4.20 *The quotient $T/\langle f \rangle$ is a finite tree.*

Fan points. To complete the proof of the structure theorem, we must show that the proposed vertex set

$$V = \overline{B}(T) \cup \overline{C}(T) \cup \overline{p}$$

is discrete. It is useful to first study the totally invariant set $T_{\mathrm{per}} \subset T$ consisting of preperiodic points for f.

Proposition 4.21 *The preperiodic points T_{per} form a closed subset of T, which projects to a finite subtree $T_{\mathrm{per}}/\langle f \rangle \subset T/\langle f \rangle$.*

Proof. Let $S \supset p$ be the finite subtree projecting isometrically onto $T/\langle f \rangle$ provided by Proposition 4.19. By Proposition 4.5, $P = T_{\mathrm{per}} \cap S$ is a closed, finite subtree of S, so the same is true of $\pi(P)$; and $T_{\mathrm{per}} = \pi^{-1}(\pi(P))$. ■

We say $y \in T$ is a *fan point* if y is preperiodic, but it is a limit of points which are not, as in Figure 3. The set $F(T)$ of all fan points clearly satisfies

$$F(T) = \overline{F}(T) \subset B(T) \subset V.$$

Corollary 4.22 *The set of fan points $F(T)$ is discrete, and $\pi(F(T))$ is a finite set including no endpoints of $T/\langle f \rangle$ except possibly $\pi(p)$.*

Proof. By the preceding result, $\pi(F(T))$ forms the boundary of the finite subtree $S = T_{\mathrm{per}}/\langle f \rangle \subset T/\langle f \rangle$, which includes p. If ∂S contains an endpoint x, then we must have $S = \{x\}$ and hence $x = \pi(p)$. ■

Proposition 4.23 *The map $\pi : T \to T/\langle f \rangle$ is a local homeomorphism outside the set $F(f) \cup \overline{C}(f) \cup \overline{p}$.*

Proof. Suppose x is preperiodic. Since f^i is a local homeomorphism at x, its images are also branch points, so we can assume x is periodic, say with period n. Since $x \notin F(f) \cup \overline{C}(f) \cup \overline{p}$, we can choose $r > 0$ such that $B_0 = B(x, r)$ consists only of preperiodic points, the balls $B_i = f^i(B_0)$ for $i = 0, 1, \ldots, n-1$ are disjoint, $f^i | B_0$ is an isometry for all i, and $p \notin B_0$.

In particular, $f^n : (B_0, x) \to (B_0, x)$ is a (bijective) ribbon automorphism. Since every $y \in B_0$ is preperiodic under f, it is actually periodic under f^n. By Proposition 3.1, every $y \in B_0 - \{x\}$ has the same period. But f^n fixes the arc $[x,p] \cap B_0$, so $f^n|B_0$ is the identity.

This shows that distinct points in B_0 lie in distinct grand orbits, and hence $\pi|B_0$ is injective, and therefore an isometry. By periodicity, any grand orbit $\overline{y}$ with $d(\overline{y}, \overline{x}) < r$ must meet B_0, so $\pi(B_0) = B(\pi(x), r)$. In particular $\pi(B_0)$ is open, and hence $\pi|B_0$ is a local homeomorphism.

Now suppose x has an infinite forward orbit. Since $\overline{x}$ is uniformly discrete and $d(\overline{x}, \overline{C}(f)) > 0$, we can choose an open ball $B_0 = B(x,r)$ such that $f^i|B_0$ is an isometry for all $i \geq 0$, and the balls $B_i = f^i(B_0)$ are disjoint. This again implies that $\pi(B_0) = B(\pi(x), r)$, and $\pi|B_0$ is an isometric local homeomorphism. ■

Corollary 4.24 *The set of branch points $B(T)$ is discrete, and $\pi(B(T))$ is finite.*

Proof. This is immediate for branch points in the discrete set $F(f) \cup \overline{C}(f) \cup \overline{p}$, and by the preceding result any other $x \in B(T)$ must map to one of the finitely many branch points of $T/\langle f \rangle$. ■

Corollary 4.25 *The set $V \subset T$ is discrete, and $\pi(V)$ is finite.*

Proposition 4.26 *The endpoints of T are contained in $\overline{C}(f) \cup \overline{p}$.*

Proof. Let $S_0 = \bigcup_{c \in C(f)} [c,p]$, let $S_i = f^i(S_0)$ and let E_i denote the endpoints of S_i. We have $E_0 = \{c, p\}$ and $E_{i+1} \subset f(E_i \cup C(f))$, so $E_i \subset \overline{C}(f) \cup \overline{p}$ for all i. Now let x be an endpoint of T. By minimality, there are $i, j > 0$ such that $f^i(x) \in S_j$. Since f is a branch covering, $f^i(x)$ is an endpoint of T for all i. Thus $f^i(x) \in E_j$ is also an endpoint of S_j, and hence $x \in \overline{C}(f) \cup \overline{p}$. ■

Corollary 4.27 *The endpoints of $T/\langle f \rangle$ are contained in $\pi(C(f) \cup \{p\})$.*

Proof. Suppose $\pi(x)$ is an endpoint of $T/\langle f \rangle$ and $x \notin \overline{C}(f) \cup \overline{p}$. Then $x \notin F(f)$ by Corollary 4.22; hence π is a local homeomorphism at x, so x is an endpoint of T. But then $x \in \overline{C}(f) \cup \overline{p}$, as we have just seen. ■

Proof of Theorem 4.1. We have seen that $T/\langle f\rangle$ is a finite tree, and that $V = \overline{B}(T) \cup \overline{C}(f) \cup \overline{p}$ is a discrete subset of T, containing all its endpoints and branch points. By minimality, V spans T, and hence T is simplicial with respect to V.

Since V contains $C(f)$ and $f^{-1}(V) = V$, $f : (T - V) \to (T - V)$ is an isometric covering map, and hence f is simplicial with respect to V. Similarly $\pi : T \to T/\langle f\rangle$ is simplicial with respect to V and $V/\langle f\rangle$, since π is a local homeomorphism outside $F(f) \cup \overline{C}(f) \cup \overline{p} \subset V$. To see that π is open at each vertex $x \in V$, consider any edge e adjacent to $\pi(x)$. We just need to show there is an edge e' adjacent to x with $\pi(e') = e$. But this is clearly true with x replaced by $f^i(x)$ for some $i > 0$; and since f is open, it is also true for x.

Let e, b and e', b' denote the number of endpoints and branch points of $T/\langle f\rangle$ and $T_{\rm per}/\langle f\rangle$ respectively. Clearly $e' \le e$ and $b' \le b \le e - 2$, and $e \le |C(f)| + 1$ by Corollary 4.27. Recalling that $F(f) - \overline{p}$ projects to the ends of S other than p, and that

$$B_0(T) = B(T) - (F(f) \cup \overline{C}(f) \cup \overline{p})$$

projects to branch points of $T/\langle f\rangle$ (by Proposition 4.23), we find:

$$\begin{aligned} |\pi(V)| &\le |\pi(C(f) \cup \{p\})| + |\pi(F(f) - \overline{p})| + |\pi(B_0(T))| \\ &\le e + (e' - 1) + b \le 3e - 3 \le 3|C(f)|. \end{aligned}$$

Finally note that $T/\langle f\rangle$ has at least $e' - 1$ ends outside of $T_{\rm per}/\langle f\rangle$, each of which is represented by an aperiodic critical orbit; this implies

$$|\pi(V \cap T_{\rm per})| \le |\pi(V)| - (e' - 1) \le e + b \le 2|C(f)|,$$

as claimed. ∎

Orbits are eventually bounded. For later use in §11, we conclude by observing the following consequences of the structure theorem.

Proposition 4.28 *There is an $R > 0$ such that the forward orbit of every point $x \in T$ eventually lands in $B(p, R)$.*

Proof. Choose R large enough that $B(p, R)$ contains at least one lift of each of the finitely many edges of $T/\langle f\rangle$. Then $B(p, R)$ meets every grand orbit of f, and is invariant under forward iteration, so it contains the tail of every forward orbit. ∎

Corollary 4.29 *Any segment $[a,b] \subset T$ of length more than $2R$ meets $\overline{C}(f)$.*

Proof. Cover $[a,b]$ by a finite number of edges of T, and choose n large enough that each maps into $B(p,R)$ under f^n. Then we have $f^n([a,b]) \subset B(p,R)$ and hence $d(f^n(a), f^n(b)) \leq 2R < d(a,b)$. This implies $[a,b]$ contains a critical point of f^n, and so it must meet $\overline{C}(f)$. ∎

5 Suspensions and cores

In this section we show that the shape of a branched covering is determined by its *core*, which records the dynamics on the post-critical set. This result allows us to define the *suspension* of a dynamical system (F,S) on S^1, which will play a central role in the theory of algebraic limits.

Cores. Let $f : (T,p) \to (T,p)$ be a minimal, degree $d \geq 2$ branched covering of a ribbon $\mathbb{R}$-tree. The *core* of f is the simplicial branched covering

$$f_1 : (S_1, p) \to (S_0, p) \subset (S_1, p)$$

where

$$S_0 = \bigcup_{n=1}^{\infty} \bigcup_{c \in C(f)} [f^n(c), p]$$

is the convex hull of p and the post-critical set of f, $S_1 = f^{-1}(S_0)$, and $f_1 = f|S_1$.

We will show that f is functorially determined by f_1, and characterize those mappings which can arise as cores.

Theorem 5.1 *Let $f_1 : (S_1, p) \to (S_0, p) \subset (S_1, p)$ be a simplicial branched covering of ribbon $\mathbb{R}$-trees with respect to a fixed vertex set. Suppose that S_0 is the convex hull of p and the forward orbits of the critical points of f_1. Then:*

1. *There is a minimal branched covering $f : (T,p) \to (T,p)$ whose core is isomorphic to f_1; and*

2. *Any isomorphism between f_1 and the core g_1 of another minimal branched covering g extends uniquely to an isomorphism between f and g.*

Proof. It is convenient to replace $f_1 : S_1 \to S_0 \subset S_1$ by a finite covering map

$$\widehat{f}_1 : \widehat{S}_1 \to \widehat{S}_0 \subset \widehat{S}_1$$

between 'blowups' $\widehat{S}_i \to S_i$ of the given simplicial trees, where each vertex v is replaced with a small oriented circle C_v, and each edge incident to v is attached to a distinct point of C_v, respecting the cyclic order. The blowup can be chosen so the diagram

$$\begin{tikzcd} \widehat{S}_1 \arrow[r, "\widehat{f}_1"] \arrow[d] & \widehat{S}_0 \arrow[d] \\ S_1 \arrow[r, "f_1"] & S_0 \end{tikzcd}$$

commutes. Since each component of $\widehat{S}_1 - \widehat{S}_0$ is attached to $\widehat{S}_0$ by a single edge, we also have a unique retraction

$$\rho_1 : \widehat{S}_1 \to \widehat{S}_0$$

which is locally constant outside $\widehat{S}_0$.

We now proceed inductively. Given ρ_i and $\widehat{f}_i$, let

$$\widehat{S}_{i+1} = \{(x,y) \in \widehat{S}_i \times \widehat{S}_i \ : \ \rho_i(x) = \widehat{f}_i(y)\},$$

and define $\rho_{i+1}, \widehat{f}_{i+1} : \widehat{S}_{i+1} \to \widehat{S}_i$ by $\rho_{i+1}(x,y) = x$ and $\widehat{f}_{i+1}(x,y) = y$. These new covering maps and retractions make the diagram

$$\begin{tikzcd} \widehat{S}_{i+1} \arrow[r, "\rho_{i+1}"] \arrow[d, "\widehat{f}_{i+1}"'] & \widehat{S}_i \arrow[d, "\widehat{f}_i"'] \\ \widehat{S}_i \arrow[r, "\rho_i"] & \widehat{S}_{i-1} \end{tikzcd}$$

commute.

Letting $i \to \infty$ and then blowing down, we obtain a sequence of simplicial ribbon $\mathbb{R}$-trees

$$(S_0, p) \subset (S_1, p) \subset (S_2, p) \subset \cdots$$

together with compatible branched coverings $f_i : S_i \to S_{i-1}$. Let $(T, p) = \bigcup (S_i, p)$ and $f = \bigcup f_i$. It is then straightforward to verify that $f : (T, p) \to (T, p)$ is a minimal branched covering whose core is f_1.

Similarly, any isomorphism between f_1 and the core g_1 of another branched covering g can be pulled back, using the dynamics, to give a unique embedding of f_i into g, and then by minimality to give an isomorphism between f and g. ∎

Suspension. An *effective divisor* $S = \sum_1^k m_i s_i$ on the circle is a formal sum of distinct points $s_i \in S^1$ with integral coefficients $m_i \geq 1$. We let $\operatorname{supp} S = \{s_1, \ldots, s_k\} \subset S^1$.

Let (F, S) be a pair consisting of a covering map $F : S^1 \to S^1$ of positive degree, and a nontrivial effective divisor S. Let $A = \bigcup_0^\infty F^n(\operatorname{supp} S)$.

Proposition 5.2 *There exists a minimal branched covering $f : (T, p) \to (T, p)$ of degree $d = \deg(F) + \deg(S)$, and an embedding $\iota : A \to T$, such that*

1. *p is a critical point of multiplicity $\deg(F) - 1$;*
2. *$\iota(s_i)$ is a critical point of multiplicity m_i, $i = 1, \ldots, k$;*
3. *$d(\iota(t), p) = 1$ for all $x \in A$; and*
4. *$\iota(F(t)) = f(\iota(t))$ for all $x \in A$.*

The pair (f, ι) is unique up to isomorphism.

We refer to the branched covering f as the *suspension* of (F, S).

Proof. We first construct the core of f. We will regard $S^1 = \mathbb{R}/\mathbb{Z}$ as the unit circle in $\mathbb{C}$, via the coordinate $z = \exp(2\pi i t)$. Let $p = 0$ and let $S_0 \subset \mathbb{C}$ denote the union of the segments $[p, a]$, $a \in A$. With its induced path metric and the given planar topology, S_0 is a simplicial ribbon $\mathbb{R}$-tree. The natural inclusion $A \subset S_0$ will play the role of ι.

Let $S_1' = \bigcup_{F(z) \in A} [p, z]$ be the cone over $F^{-1}(A)$, again with its natural ribbon structure. To construct S_1, attach m_i copies of S_0 to S_1' at each point $s_i \in \operatorname{supp} S$, gluing $F(s_i)$ in S_0 to $s_i \in S_1$. Let $f_1 : (S_1, p) \to (S_0, p)$ be the map that sends each attached copy to the original S_0 by the identity, and sends $[p, a]$ isometrically to $[p, F(a)]$ for each $a \in F^{-1}(A)$.

By construction, f_1 has critical points of multiplicity m_i at each point s_i, and a critical point of multiplicity $\deg(F) - 1$ at p. Since S_0 is simply the cone over the forward orbits of the points s_i, Theorem 5.1 implies f_1 extends to a minimal branched covering $f : (T, p) \to (T, p)$ of degree $\deg(F) + \deg(S)$. Conversely, any other branched covering related to (F, S) as above contains f_1 as its core, so it is isomorphic to f. ■

Note that a topological conjugacy between (F_1, S_1) and (F_2, S_2) determines an *isometric* conjugacy between their suspensions.

6 Ends and markings

Let $f : (T, p) \to (T, p)$ be a minimal branched covering with $\deg(f) \geq 2$. By Proposition 3.2, there is an induced covering map

$$f_* : \epsilon(T) \to \epsilon(T)$$

on the cyclically ordered set of ends of T.

In this section we show that T almost always has infinitely many ends, and that there is a natural order-preserving *marking*

$$\phi : \epsilon(T) \to S^1$$

that transports the dynamics of f_* on $\epsilon(T)$ to the dynamics of $p_d(t) = d \cdot t \bmod 1$ on S^1. This marking is well-defined up to post-composition with an element of the automorphism group $\mathbb{Z}/(d-1)$ of $p_d(t)$.

The cone over z^d. To begin we discuss a simple, exceptional example. Fix $d \geq 2$, and let $\mathbb{Z}[1/d] \subset \mathbb{Q}$ be the dense subring generated by $1/d$. Let $E_d = \mathbb{Z}[1/d]/\mathbb{Z}$ denote its projection to S^1, and let $p = 0 \in E_d$.

Let T_d be the $\mathbb{R}$-tree obtained as the unit cone over E_d based at a new vertex c_0. By definition, T_d is a union of unit edges $[c_0, e]$, $e \in E_d$, meeting only at c_0. The cyclic ordering on E_d gives T_d a ribbon structure, and the covering map $F_d : E_d \to E_d$ given by $F_d(x) = dx \bmod 1$ extends uniquely to a branched covering $F_d : (T_d, p) \to (T_d, p)$. The map F_d has a unique critical point, c_0, which is also a fixed point.

We say $f : (T, p) \to (T, p)$ is *a cone over* z^d if it is isomorphic to $F_d : (T_d, p) \to (T_d, p)$, up to rescaling the metric on T_d by a constant. It is easy to verify:

Proposition 6.1 *A degree d, minimal branched covering $f : (T, p) \to (T, p)$ is a cone over z^d if and only if it has a totally invariant critical point.*

Dynamics on ends. Note that the tree T_d for a cone over z^d is bounded; in particular, $\epsilon(T_d) = \emptyset$. These are the only such examples. More precisely, we will show:

Theorem 6.2 *Let $f : (T, p) \to (T, p)$ be a minimal branched covering of degree $d \geq 2$. Then either:*

1. *$\epsilon(T)$ is empty, and f is a cone over z^d; or*

2. *$f_* | \epsilon(T)$ has a fixed-point, or*

3. *$f_*|\epsilon(T)$ has a point of period two, but no fixed point, and every critical point of f is fixed.*

An example where (3) arises will be given in §8.

Constructions of rays. To begin the proof of Theorem 6.2, recall that f is simplicial with respect to the vertex set $V = \overline{C}(f) \cup \overline{B}(f) \cup \overline{p}$. Suppose $x \in V$, $f(x) \neq x$ and

$$[x, f(x)] = [x, v_1] \cup [v_1, v_2] \cup \cdots \cup [v_n, f(x)] \tag{6.1}$$

expresses $[x, f(x)]$ as a union of consecutive edges with respect to the vertex set V.

Lemma 6.3 *If $f(v_1) \neq v_n$, then $f_*|\epsilon(T)$ has a fixed point.*

Proof. Since f is open, there is a simplicial edge $[w, x]$ with $f(w) = v_n$. By hypothesis, $w \neq v_1$. Let $\alpha_0 = [x, f(x)]$. Since f is open, there is an arc α_1 mapping bijectively to α_0. We can construct the lift α_1 by first lifting $[v_n, f(x)]$ to $[w, x]$, and then inductively lifting the remaining edges of α_0. The result need not be unique, but by construction it satisfies $\alpha_0 \cap \alpha_1 = \{x\}$.

Repeating the construction, we obtain a chain of finite arcs $\alpha_0, \alpha_1, \alpha_2 \ldots$ meeting only at adjacent endpoints, such that $f(\alpha_i) = \alpha_{i-1}$. Then the ray $\alpha = \bigcup_1^\infty \alpha_i$ based at $f(x)$ satisfies $f(\alpha) = \alpha_0 \cup \alpha$, and thus α represents an end of T fixed by f_*. ∎

Corollary 6.4 *Either $f_*|\epsilon(T)$ has a fixed point, or every critical point of f is fixed.*

Proof. Suppose $x \in C(f)$ and $x \neq f(x)$. We must show f_* has a fixed point. Express $[x, f(x)]$ as in (6.1). By the lemma above, we may assume $f(v_1) = v_n$. Since x is a critical point, there is a simplicial edge $[x, w] \neq [x, v_1]$ with the same image as $[x, v_1]$ under f. Then we have

$$[w, f(w)] = [w, x] \cup [x, v_1] \cup \cdots [v_{n-1}, v_n];$$

and since $f(x) \neq v_{n-1}$, the lemma again yields a fixed point for f_*. ∎

Lemma 6.5 *If f has two fixed critical points, then $f_*|\epsilon(T)$ has a point of period two.*

Proof. Let c_0, d_0 be pair of distinct critical points that are fixed by f. Then the arc $[c_0, d_0]$ is also fixed. Since f branches at c_0 and d_0, there are additional arcs $[d_1, c_0]$ and $[d_0, c_1]$ that each map isometrically to $[c_0, d_0]$ and meet it only at their endpoints. Taking further preimages, we obtain a chain of adjacent arcs

$$\ldots [c_2, d_1] \cup [d_1, c_0] \cup [c_0, d_0] \cup [d_0, c_1] \cup [c_1, d_2] \ldots$$

such that $f(c_i) = c_{i-1}$ and $f(d_i) = d_{i-1}$. The ends of this chain determine a pair of points in $\epsilon(T)$ that are exchanged by f_*. ■

Proof of Theorem 6.2. If every critical point of f is fixed, then either $|C(f)| = 1$ — in which case $\epsilon(T)$ is empty and f is isomorphic to F_d; or $|C(f)| > 1$ — in which case $f_*|\epsilon(T)$ has a point of period two, by the preceding lemma. Otherwise, f_* has a fixed point by Corollary 6.4. ■

7 Length functions

Let $f : (T, p) \to (T, p)$ be a minimal branched covering of degree $d \geq 2$. In this section we introduce the *length function*

$$Lf : \epsilon(T) \to \mathbb{R}$$

to measure the rate at which f moves points of T towards p; it is defined by

$$Lf(\alpha) = \lim_{x \to \alpha} d(x, p) - d(f(x), p).$$

Let

$$M(f) = \max\{d(x, p) \ : \ f(x) = p\}.$$

We will show:

Theorem 7.1 *For any minimal branched covering $f : (T, p) \to (T, p)$:*

1. *The map $Lf : \epsilon(T) \to \mathbb{R}$ is a piecewise constant function, with at most $(e - 1)$ laps, where $e = |f^{-1}(p)| \leq d$;*

2. *The values of Lf are contained in the group $\oplus_1^n \mathbb{Z}\ell_i$ generated by the lengths of the edges of $T/\langle f \rangle$; and*

3. *We have* $0 \le Lf(\alpha) \le M(f)$ *for all ends* α.

The bound on the lap number in (1) means that $Lf(\alpha)$ rises and falls at most $e-1$ times as α moves once around $\epsilon(T)$.

We will also discuss the upper and lower envelopes $\underline{L}f, \overline{L}f : S^1 \to [0, M(f)]$.

Reversed edges. Let $r(x) = d(x,p)$ as in §4, and define

$$\tau(x) = r(x) - r(f(x)) \ge 0.$$

Recall that $f : (T,p) \to (T,p)$ is a simplicial map with respect to the vertex set $V = \overline{C}(f) \cup B(T) \cup \overline{p}$. Every edge $e = [v_1, v_2] \subset T$ has a preferred orientation, namely the one where $r(v_1) < r(v_2)$. That is, we think of the edges of T as being directed away from p.

We say e is a *reversed edge* if $f|e$ is orientation-reversing; that is, if $r(f(v_1)) > r(f(v_2))$. We let

$$R(f) = \bigcup \{e \subset T \ : \ e \text{ is a reversed edge}\}.$$

Evidently we have

$$\tau(v_2) = \begin{cases} \tau(v_1) + 2d(v_1, v_2) & \text{if } e = [v_1, v_2] \text{ is reversed, and} \\ \tau(v_1) & \text{otherwise;} \end{cases}$$

and thus

$$\tau(x) = 2 \operatorname{length}([x,p] \cap R(f)).$$

Proposition 7.2 *Any reversed edge of* T *is contained in an arc of the form* $[v,p]$ *where* $f(v) = p$. *Conversely, any vertex* $v \neq p$ *with* $f(v) = p$ *is an endpoint of a reversed edge.*

Proof. Let $e = [v_1, v_2]$ be a reversed edge. If $f(v_2) \neq p$, then since f is open there is an adjacent reversed edge $[v_2, v_3]$ with $r(v_1) < r(v_2) < r(v_3)$ and $r(f(v_1)) > r(f(v_2)) > r(f(v_3))$. Applying the same reasoning to $[v_2, v_3]$, we eventually reach a vertex v_n with $[v_1, v_2] \subset [p, v_n]$ and $f(v_n) = p$.

For the converse, note that if $v \neq p$ and $f(v) = p$ then the unique edge $[v,w]$ with $r(w) < r(v)$ is reversed by f. ■

Corollary 7.3 *The critical points of* f *are contained in the finite subtree* $\bigcup_{f(x)=p} [x,p]$.

Proof. Every critical point $c \neq p$ is incident to a reversed edge. ■

Corollary 7.4 *The reversed edges, together with p, span a finite subtree whose endpoints lie in $f^{-1}(p)$.*

Corollary 7.5 *We have* $\max\{\tau(x) \,:\, x \in T\} = \max\{d(v,p) \,:\, f(v) = p\}$.

Translation lengths of ends. We define the *length function* $Lf : \epsilon(T) \to [0, \infty)$ on the ends of T by

$$Lf(\alpha) = \lim_{x \to \alpha} \tau(x).$$

Equivalently, if α is a ray based at p then

$$Lf(\alpha) = 2 \operatorname{length}(\alpha \cap R(f)), \tag{7.1}$$

where $R(f)$ is the union of the reversed edges of f.

Proof of Theorem 7.1. By Corollary 7.5, we have $0 \le Lf(\alpha) \le M(f)$. Let us scale the metric so $M(f) = 1$, and represent the ends of T by rays based at p. Let $S_0 = \bigcup_{f(x)=p} [x, p]$, and let $\pi : \epsilon(T) \to S_0 \cap V$ be the nearest-point projection, characterized by

$$[\pi(\alpha), p] = \alpha \cap S_0.$$

Then $Lf(\alpha) = \tau(\pi(\alpha))$.

Because of the ribbon structure, we can partition $\epsilon(T)$ into a cyclically ordered set of disjoint intervals $(I_1, \ldots I_m)$ such that π maps I_i to a single vertex v_i, and adjacent vertices in the cyclically ordered list $(v_1, \ldots, v_m)$ are distinct. (Each vertex of degree k occurs k times in this list; in particular, m is twice the number of edges in S_0.)

Since m is finite, Lf is piecewise constant. The local maxima of the cyclically ordered sequence $(\tau(v_1), \ldots, \tau(v_m))$ occur at the $e - 1$ vertices $v_i \neq p$ with $f(v_i) = p$, so Lf has at most $e - 1$ laps. Finally the values of Lf are finite sums of lengths of edges of T, so they lie in the group generated by the edge lengths of $T/\langle f \rangle$. ■

Envelopes. Now assume f is not a cone over z^d; equivalently, $\epsilon(T) \neq \emptyset$. Then the image of $\phi : \epsilon(T) \to S^1$ is dense in the circle (since it is invariant under p_d^{-1}).

It is convenient to transport the information recorded by Lf from $\epsilon(T)$ to S^1 using the marking ϕ. For this purpose, given $t \in S^1$, we define the upper and lower envelopes of Lf by:

$$\begin{aligned} \overline{L}f(t) &= \lim_{h \to 0} \sup\{Lf(\alpha) \,:\, \phi(\alpha) \in [t-h, t+h]\}, \quad \text{and} \\ \underline{L}f(t) &= \lim_{h \to 0} \inf\{Lf(\alpha) \,:\, \phi(\alpha) \in [t-h, t+h]\}. \end{aligned}$$

Clearly $\overline{L}f(t)$ and $\underline{L}f(t)$ are upper and lower semicontinuous respectively, and $\underline{L}f(t) \le \overline{L}f(t)$. Theorem 7.1 implies:

Corollary 7.6 *The upper and lower envelopes $\overline{L}f, \underline{L}f : S^1 \to [0,1]$ are piecewise constant functions with at most $(d-1)$ laps, and $\overline{L}f(t) = \underline{L}f(t)$ at any point where both are continuous.*

8 Quadratic trees

In this section we turn to examples of degree two, and establish:

Theorem 8.1 *A normalized minimal branched covering $f : (T,p) \to (T,p)$ of degree two is determined up to isomorphism by its rotation number $\rho(f) \in S^1$ and, when $\rho(f)$ is rational, its fan height $\delta(f) \in [-1,1]$.*

Moreover, every possible pair of invariants arises for some f.

This result shows the moduli space of quadratic trees forms a topological circle, obtained from S^1 by inserting a copy of $[-1,1]$ at each rational point.

We then give two results that describe the length functions on these quadratic trees in terms of combinatorially defined intervals $K(t)$ and $K_\pm(a/b) \subset S^1$. As in §7, $M(f) = \max\{d(x,p) : f(x) = p\}$.

Theorem 8.2 *If $\rho(f) = a/b$ is a nonzero rational, then*

$$\frac{\overline{L}f(t)}{M(f)} = \begin{cases} 1 & \text{if } t \in K(a/b), \\ |\delta(f)| & \text{if } t \notin K(a/b) \text{ but } t \in K_\epsilon(a/b), \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

Here $\epsilon = \pm$ is chosen to agree with the sign of $\delta(f)$.

Theorem 8.3 *If $\rho(f)$ is irrational, then $\overline{L}f(t)/M(f) = 1$ for $t \in K(\rho(f))$ and $\overline{L}f(t) = 0$ otherwise.*

Remarks. The lower envelope is almost the same as the upper envelope; by Corollary 7.6, it is given by $\underline{L}f(t) = \liminf_{s\to t} \overline{L}f(t)$.

The simple form of $\overline{L}f$ above results from the fact that a quadratic tree can have at most 2 reversed edges. In contrast, we will see at the end of this section that a cubic tree can have arbitrarily many reversed edges, and its length function $\overline{L}f$ can assume arbitrarily many distinct values.

Another formulation of these results, applicable in higher degree but excluding the case $\delta(f) \neq 0$, will be presented in §13.

Rotation number and fan height. We begin with some definitions. Let $f : (T, p) \to (T, p)$ be a minimal quadratic branched covering, and let c_0 denote its unique critical point. Let $c_n = f^i(c_0)$, and let $P(f) = \{c_1, c_2, c_3 \ldots\}$ denote the post-critical set. Observe that $d(c_n, p) = d(c_0, p) > 0$ for all $n > 0$; thus the ribbon structure on the bouquet $\bigcup_0^\infty [c_n, p]$ determines a cyclic ordering on $P(f)$. We say f is *normalized* if $d(c_0, p) = 1$.

The *rotation number* $\rho(f)$ is defined, using equation (2.4), to be the rotation number of $f : P(f) \to P(f)$. When $\rho(f) = a/b$ is rational, we also define the *fan height* of f by

$$\delta(f) = \pm \frac{d(c_1, c_{b+1})}{d(c_0, p)}.$$

Here the sign is chosen to be positive if $c_1 < c_{b+1} < c_{2b+1}$ in the cyclic ordering on $P(f)$, and otherwise negative.

Construction of examples. Next we describe the quadratic examples with rotation number $t \in S^1$ and $\delta(f) = 0$. In the sense of Proposition 5.2, these examples are simply the suspensions of pairs (F, S) with $F(x) = x + t \bmod 1$ and $\deg(S) = 1$.

To describe them directly, let $c_1 = \exp(2\pi i t)$ and let $c_n = c_1^n$. Let $p = 0$ and let

$$S_0 = \bigcup_1^\infty [p, c_n] \subset \mathbb{C}$$

be the cone over the positive powers of c_1. With its induced path metric and planar embedding, (S_0, p) is a ribbon $\mathbb{R}$-tree and $d(p, c_1) = 1$.

Let $f_0 : (S_0, p) \to (S_0, p)$ be the unique simplicial map satisfying $f_0(c_n) = c_{n+1}$. Let $f_1 : S_1 \to S_0$ be a 2-fold covering of S_0 branched over c_1. Since c_1 is an endpoint of S_0, we can lift f_0 to an embedding $i : S_0 \to S_1$ satisfying $f_1 \circ i(x) = f_0(x)$. In this way S_0 becomes a subtree of S_1, and f_1 becomes a branched covering

$$f_1 : (S_1, p) \to (S_0, p) \subset (S_1, p).$$

By Theorem 5.1, f_1 extends to a unique minimal quadratic branched covering $f : (T, p) \to (T, p)$, and by construction we have $\rho(f) = t$.

The quotient tree $T/\langle f \rangle$ is an interval, the projection of $[c_1, p]$ to $T/\langle f \rangle$ is an isometry, and the vertex set V is simply $\overline{C}(f) \cup \overline{p}$.

When $t = a/b$ is rational, we have $c_1 = c_{b+1}$ and thus $\delta(f) = 0$. In this case T has degree b at all vertices in $\overline{p}$ and infinite degree at all vertices in $\overline{C}(f)$.

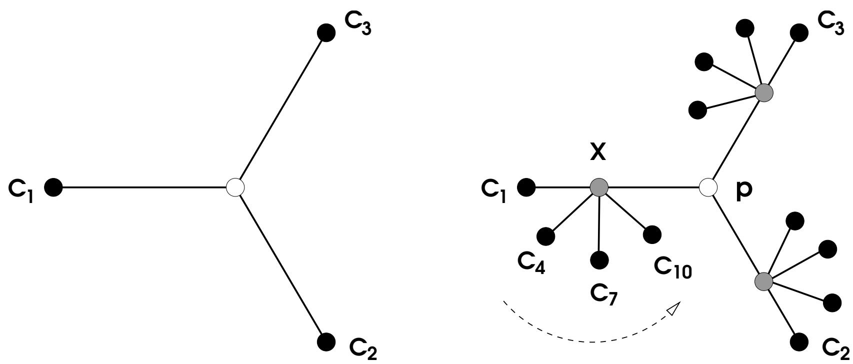


Figure 5. Trees with rotation number $1/3$: $\delta(f) = 0$ and $\delta(f) > 0$.

When t is irrational, T has infinite degree at all vertices in $\overline{p}$, degree two at c_0 and its preimages, and degree one at c_n for $n > 0$.

Examples with fans. The tree S_0 for $a/b = 1/3$ is shown at the left in Figure 5. To make examples with the same rotation number but with $\delta(f) = s \neq 0$, we start with edges $[c_1, p], \ldots, [c_b, p]$ in the same order as before, but add new points c_i, $i > b$ with

$$[c_1, p] \cap [c_{b+1}, p] = [x, p]$$

and $d(c_1, x) = s$. (When $s = 1$, $x = p$.) We further arrange that $[c_1, x]$, $[c_{b+1}, x]$, ,$[c_{2b+1}, x] \ldots$ are consecutive edges forming a positive fan around x (as shown) when $s > 0$, and a negative fan when $s < 0$. Finally we place copies of this fan on the edges $[c_2, p], \ldots, [c_b, p]$.

The resulting tree (show at the right in Figure 5) admits a unique simplicial map $f_0 : (S_0, p) \to (S_0, p)$ sending c_i to c_{i+1}. As before, there is a unique extension of f_0 to a minimal quadratic branched covering $f : (T, p) \to (T, p)$ with a critical value at c_1. By construction we have $\rho(f) = a/b$ and $\delta(f) = s$.

The vertex set for T is $V = \overline{C}(f) \cup \overline{x} \cup \overline{p}$. Note that x is a fan point of period b (if $x \neq p$). The tree T has degree b at its vertices in $\overline{p}$, and infinite degree at the rest. The quotient $T/\langle f \rangle$ is a unit interval with endpoints $\pi(c_0)$, $\pi(p)$, and a distinguished internal point $\pi(x)$.

Proof of Theorem 8.1. Let $f : (T, p) \to (T, p)$ be a normalized minimal quadratic branched cover, let $S_0 \subset T$ be the convex hull of $P(f) \cup \{p\}$, let $S_1 = f^{-1}(S_0)$ and let $f_i = f|S_i$.

If $\rho(f)$ is irrational, then the edges $[c_i, p]$ can only meet at p. The same is true if $\rho(f) = a/b$ is rational and $c_1 = c_{b+1}$. In either case, $f_0 : S_0 \to S_0$ is isomorphic to the cone over an orbit of the rotation $x \mapsto x + \rho(f)$ on S^1.

Now suppose $\rho(f) = a/b$ is rational but $c_{b+1} \neq c_1$. Then $\rho(f^b|P(f) = 0)$. It follows that the points c_1 and c_{b+1} are consecutive in the cyclic ordering on $P(f)$. Then $[c_1, p] \cap [c_{b+1}, p] = [x, p]$ for some x, and $f^b(x) = x$. Since $f|P(f)$ preserves the cyclic ordering, the points $c_1, c_{b+1}, c_{2b+1}, \ldots$ form a fan about x as in Figure 5, and the structure of $f_0 : S_0 \to S_0$ is again uniquely determined.

Since there is a unique 2-fold cover of S_0 branched over c_1, the map f_0 determines the core $f_1 : (S_1, p) \to (S_0, p)$, which in turn determines f by Theorem 5.1. Thus shows f is determined by $\rho(f)$ and $\delta(f)$, and we have seen explicitly that all possible pairs of invariants arise. ■

Combinatorics of the doubling map. Next we define the intervals on S^1 that will be used to describe the length function Lf.

Let $p_2 : S^1 \to S^1$ be the doubling map $t \mapsto 2t \bmod 1$. Given a rational number $a/b \bmod 1$, there is a unique periodic cycle $C(a/b)$ such that $p_2|C(a/b)$ has rotation number a/b (see e.g. [Gol], [Mc4]). The b points of $C(a/b)$ partition the circle into closed intervals $I_0(a/b), \ldots I_{b-1}(a/b)$ with

$$|I_i(a/b)| = \frac{2^i}{2^b - 1}$$

such that $p_2(I_{i-1}(a/b)) = I_i(a/b)$ for $0 < i < b$.

On the other hand p_2 maps $I_{b-1}(a/b)$ onto the whole circle, with the points in $I_0(a/b)$ covered twice. More precisely, we can write $I_{b-1}(a/b)$ as a union of three consecutive intervals with disjoint interiors,

$$I_{b-1}(a/b) = K_-(a/b) \cup K(a/b) \cup K_+(a/b),$$

such that

$$p_2 : K(a/b) \to I_1(a/b) \cup \cdots \cup I_{b-1}(a/b)$$

is a bijection, while the map

$$p_2 : K_-(a/b) \cup K_+(a/b) \to I_0(a/b)$$

is a bijection on each of the two subintervals $K_\pm(a/b)$.

Example. When $a/b = 2/5$ we have $C(a/b) = (5, 10, 20, 9, 18)/31$, and the endpoints of the consecutive intervals $K_-(a/b), K(a/b)$ and $K_+(a/b)$ are $t = (40, 41, 9, 10)/62$. These values of t are the angles (mod 1) of the external rays for $P(z) = z^2 + c(2/5)$ which land at α and $-\alpha$, adjacent to the immediate basin of $z = 0$ (see Figure 1 in the Introduction).

A simple algorithms for computing the endpoints of $K(a/b)$ is given in [Mc4, §2].

Irrational rotation numbers. It is straightforward to check that as a/b approaches an irrational value $t \in S^1$, $K(a/b)$ converges to a well-defined interval $K(t) = [\theta_-(t), \theta_+(t)]$ of length $1/2$. In fact $2\theta_-(t) = 2\theta_+(t) \bmod 1$ is simply the external angle of the point $c(t)$ in the main cardioid of the Mandelbrot set where the α fixed point of $P_t(z) = z^2 + c(t)$ has multiplier $\exp(2\pi i t)$.

Proof of Theorems 8.2 and 8.3. First assume $\rho(f) = a/b$ is rational. Let $c_0 \in T$ denote the unique critical point of f, let $c_i = f^i(c_0)$ and let $f^{-1}(p) = \{p, p'\}$. Let c_{-i} denote the preimages of c_0 with $d(c_{-i}, p) = d(c_0, p)$.

Given $x \neq p$ in T, let $J(x) \subset \epsilon(T)$ (the *shadow* of x) denote the interval of ends represented by rays based at p that pass through x.

First assume $\delta(f) = 0$. Then $c_b = c_0$, and the edges $[p, c_1], \ldots, [p, c_b]$ cover a neighborhood of p. The intervals $J(c_1), \ldots, J(c_b)$ form a partition of $\epsilon(T)$ with the same cyclic ordering as $(c_1, \ldots, c_b)$, and satisfy $f_*(J(c_i)) = J(c_{i+1})$. Using the fact that the marking is a semiconjugacy, it follows easily that $\overline{\phi(J(c_i))} = I_{i-1}(a/b)$ for $i = 1, 2, \ldots, b$.

The tree T has a unique reversed edge, namely $[p', c_0]$. Thus $Lf(\alpha) = 1$ if $\alpha \in J(p')$, and $Lf(\alpha) = 0$ otherwise. It is easy to see that $J(p') \subset J(c_0)$ satisfies $f_*(J(p')) = J(c_2) \cup \cdots \cup J(c_b)$; thus $\overline{\phi(J(p'))} = K(a/b)$, and the Theorem follows.

Now suppose $\delta(f) \neq 0$; for concreteness, assume $\delta(f) > 0$. Let x_0 denote the midpoint of the arc $[c_0, c_b]$. Then $x_i = f^i(x_0)$ is a fan point of T for all $i \geq 0$, and $x_b = x_0$ (see Figure 3 for the case $a/b = 1/3$).

Let $f^{-1}(x_0) = \{x_0, x_0'\}$. Then $[p, p']$ is a union of the consecutive subintervals $[p, x_0], [x_0, c_0], [c_0, x_0'], [x_0', p']$, symmetric about c_0, with

$$d(x_0, c_0)/d(p, c_0) = \delta(f).$$

The tree T now has two reversed edges, namely $[c_0, x_0']$ and $[x_0', p']$. Thus $Lf(\alpha) = 1$ for $\alpha \in J(p')$, $Lf(\alpha) = \delta(f)$ for $\alpha \in J(x_0') - J(p')$, and $Lf(\alpha) = 0$ otherwise.

To complete the proof, write $J(x_0)$ as a union of consecutive, disjoint intervals

$$J(x_0) = I_- \cup J(p') \cup I_+.$$

Then $J(x_0') = J(p') \cup I_+$ by our assumption that $\delta(f) > 0$. It is easy to see that $f_*(I_-) = f_*(I_+) = J(x_1)$, and therefore $\overline{\phi(J(p'))} = K(a/b)$ and $\overline{\phi(I_\pm)} = K_\pm(a/b)$. Consequently $\overline{L}f(t)$ has the form given by equation (8.1), completing the proof of Theorem 8.2.

When $\rho(f) = t$ is irrational, the rays α in T passing through c_0 correspond to external angles in $K(t)$, and Theorem 8.3 follows. ■

Rotation number 0. We have excluded the case where $a/b = 0$ and $\delta(f) = 0$, since this gives the cone over z^2. When $a/b = 0$ and $\delta(f) \neq 0$, one can check that

$$\frac{\overline{L}f(t)}{M(f)} = \begin{cases} |\delta(f)| & \text{if } t \in K_\epsilon(0), \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where $K_-(0) = [0, 1/2]$ and $K_+(0) = [1/2, 0]$. (In these cases p' is an endpoint of T, so it does not define an end with $Lf(\alpha) = 1$.) These examples show one can have $\sup Lf(\alpha) < M(f)$.

Example: Cubic trees with many reversed edges. As we have just seen, a quadratic tree has at most two reversed edges. We conclude by showing that a cubic tree can have arbitrarily many.

Proposition 8.4 *For any $k > 0$ there exists a minimal cubic branched covering $f : (T, p) \to (T, p)$ such that T has more than k reversed edges, and Lf assumes more than k distinct values.*

Proof. Start with a normalized quadratic branched covering $f_0 : (T_0, p) \to (T_0, p)$ with a critical point c_0 of period $b > 1$. Choose a point x in the inverse orbit of c_0 with $d(x, p) = d(f_0(x), p) = k \gg 0$. Let $T_1 \subset T_0$ be the convex hull of p and the forward orbit of x. We can then change x into a simple critical point, and construct a cubic branched cover $f : (T, p) \to (T, p)$ which contains a copy of $f_0|T_1$. By considering the preimage of the arc $[f(x), p]$, one finds that T contains k reversed edges joining x to a preimage p' of p satisfying $d(p, p') = 2k$, and Lf assumes at least $k + 1$ different values, namely $0, 1/k, 2/k, \ldots, 1$. ■

9 Julia sets and laminations

In this section we establish:

Theorem 9.1 *The full Hubbard tree for the hyperbolic center $c(a/b)$ in the a/b-limb of the Mandelbrot set gives a quadratic branched covering $f : (T, p) \to (T, p)$ with $\rho(f) = a/b$ and $\delta(f) = 0$.*

We also discuss laminations, and use Hubbard trees to give examples of phenomena that can arise with the ends of trees.

Hubbard trees. For any rational $a/b \neq 0$ in S^1 there is a unique complex number $c(a/b)$ in the a/b limb of the Mandelbrot set such that $P(z) = z^2 + c(a/b)$ satisfies $P^b(0) = 0$ (see e.g. [Mil2], [GM, App. C].) Let

$$K(P) = \{z \in \mathbb{C} \, : \, \sup |P^n(z)| < \infty\}$$

denote the filled Julia set of P, and let $\Omega(P)$ denote its interior. There is a unique fixed point α of P separating $K(P)$ into b components, which are locally permuted with rotation number a/b.

Let $V \subset \mathbb{C}$ denote the union of the grand orbits of $z = \alpha$ and $z = 0$ under P. Let T be the simplicial tree with vertex set V, and with an edge of length one connecting $x \in \overline{\alpha}$ to $y \in \overline{0}$ whenever x lies in the boundary of the component $\Omega(P, y)$ of $\Omega(P)$ containing y. Each edge of T can be realized uniquely by a hyperbolic geodesic in $\Omega(P, y)$. The resulting planar embedding gives T a ribbon structure.

Define $f : V \to V$ by $f(x) = P(x)$. Since P sends $\Omega(P, y)$ to $\Omega(P, P(y))$, f extends uniquely to a simplicial map $f : T \to T$; and since P is a quadratic branched covering, so is f. The map f has a unique fixed point $p = \alpha$ and a unique critical point $c_0 = 0 \in V$.

We refer to T as the *full Hubbard tree* of P. (The traditional Hubbard tree is the convex hull of the postcritical set of f inside T; cf. [DH, Exposé IV], [Po]).

Proof of Theorem 9.1. By the definition of the a/b limb, the b components of $\Omega(P)$ meeting at α are cyclically permuted by P with rotation number a/b, so $\rho(f) = a/b$ as well. Since the critical point of P is periodic, $\delta(f) = 0$. ■

Siegel disks. A similar result holds for certain irrational $t \in S^1$: if the Julia set of $P(z) = \exp(2\pi i t)z + z^2$ is locally connected (as it is, for example, when t is the golden mean [Pet]), then its full Hubbard tree gives a quadratic branched covering $f : (T, p) \to (T, p)$ with $\rho(f) = t$. In these irrational cases, $P(z)$ has a Siegel disk.

Laminations. The full Hubbard tree can also be constructed purely combinatorially, using laminations.

Let us identify $S^1 = \mathbb{R}/\mathbb{Z}$ with the boundary of the unit disk $\Delta = \{z \in \mathbb{C} \, : \, |z| < 1\}$ using the coordinate $z(t) = \exp(2\pi i t)$. There is a unique hyperbolic geodesic $\gamma(t_1, t_2) \subset \Delta$ joining any pair of distinct points $z(t_1), z(t_2) \in \partial\Delta$.

A *lamination* is a closed set $\lambda \subset \Delta$ which can be expressed as a union of hyperbolic geodesics in a unique way. The *dual* of λ is the 1-complex with a vertex for each component of $\Delta - \lambda$, and an edge for each leaf of λ bordering two components.

As in §8, let $C(a/b) \subset S^1$ denote the unique periodic cycle for $p_2(t) = 2t \bmod 1$ with rotation number a/b. Let $\lambda(a/b) \subset \Delta$ denote the smallest saturated lamination containing the sides of the ideal polygon with vertices $C(a/b)$.

Theorem 9.2 *The dual of the lamination $\lambda(a/b) \subset \Delta$ is homeomorphic to the unique quadratic tree with $\rho(f) = a/b$ and $\delta(f) = 0$.*

Proof. The points $t \in C(a/b)$ are nothing more than the angles of external rays landing at the α fixed point for $P(z) = z^2 + c(a/b)$ [GM]. The lamination λ itself is a union of ideal polygons, each of which corresponds to external rays landing at a given point in the grand orbit of α. The components of λ with infinitely many sides correspond to the components of the interior $\Omega(P)$ of $K(P)$. Thus the dual of $\lambda(a/b)$ is homeomorphic to the full Hubbard tree for P, and we can apply Theorem 9.1. ∎

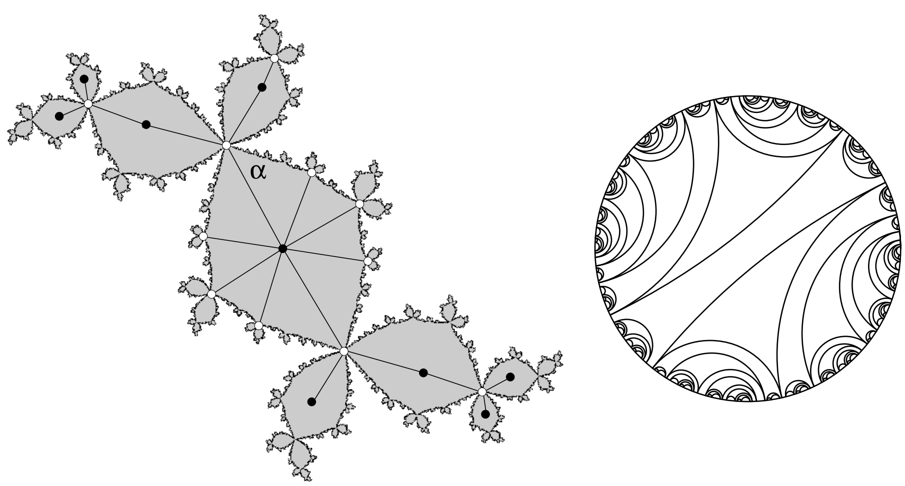


Figure 6. Hubbard tree for $a/b = 1/3$, and its dual lamination.

Example: rotation number 1/3. Figure 6 shows a portion of the full Hubbard tree T embedded in *Douady's rabbit*, the filled Julia set for $P(z) =$

$z^2 + c(1/3)$. The lamination $\lambda(1/3)$, and part of its dual tree, are shown at the right in Figure 6. Here the cycle $C(1/3) = (1,2,4)/7$ forms the vertices of an ideal triangle.

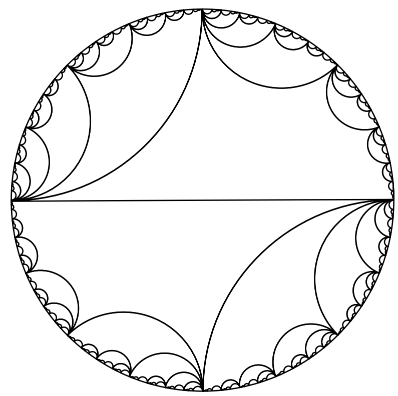


Figure 7. Limit of $\lambda(1/n)$ as $n \to \infty$.

Limitations of laminations. The definition of $\lambda(t)$ can be easily extended to the case where t is irrational, to obtain combinatorial models for all quadratic trees with $\delta(f) = 0$. However it is more difficult to use laminations to model the trees with $\delta(f) \neq 0$.

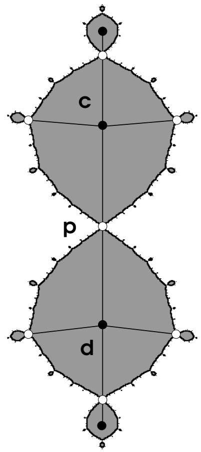
For example, the quadratic tree $f : (T, p) \to (T, p)$ with $\rho(f) = 0$ and $\delta(f) = 1$ can be thought of as the combinatorial limit of the trees $f_n : (S_n, p) \to (S_n, p)$ with $\delta(f_n) = 0$ and $\rho(f_n) = 1/n$. The tree S_n in turn is dual to the lamination $\lambda(1/n)$, which converges geometrically to the lamination λ shown in Figure 7.

But the dual of λ is only part of the tree T; half of each fan is missing. A problem arises because the endpoints of the leaf $\gamma(0, 1/2) \subset \lambda$ are identified by $s(t)$, so its forward orbit disappears. It is to preserve this part of the dynamics that we have chosen to focus our attention on trees.

Example: A cubic tree with no invariant end. We can now show that case (3) of Theorem 6.2, where $f_*|\epsilon(T)$ has a cycle of period two but no fixed points, actually occurs.

Consider the full Hubbard tree $f : T \to T$ for the cubic polynomial $P(z) = (z^3 + 3z)/2$. The map P has a pair of fixed critical points at $z = \pm i$, so the same is true for f (see Figure 8).

We claim $f_*|\epsilon(T)$ has no fixed ends. Indeed, a fixed end of T would give a fixed point in $J(P)$ which is not on the boundary of any component



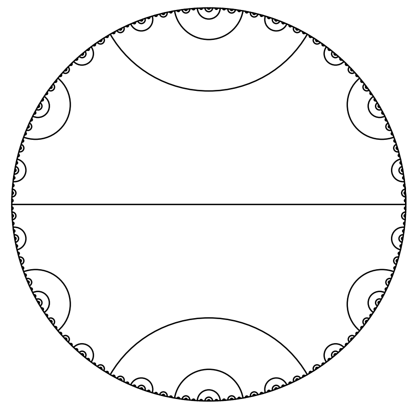


Figure 8. The full Hubbard tree for $P(z) = (z^3 + 3z)/2$ has no fixed end; the corresponding lamination contains the leaf $[-1, 1]$ connecting the fixed points of $z \mapsto z^3$ on S^1.

of $\Omega(P) = \operatorname{int} K(P)$; but the only fixed point in $J(P)$ is $z = 0$, which lies on the boundary of the immediate basins for both critical points of P (and corresponds to the midpoint of the edge $[c_0, d_0]$ connecting the critical points of T).

On the other hand the tips of $J(P)$ at $z = \pm i\sqrt{5}$ form a cycle of period two, corresponding to the pair of ends of T exchanged by f_* that were constructed in Lemma 6.5.

Example: Quadratic trees with missing periodic ends. In a similar spirit, consider the quadratic tree $f : (T, p) \to (T, p)$ with rotation number $\rho(f) = a/b$ and $\delta(f) = 0$. Let $C \subset S^1$ be the unique cycle with rotation number a/b. Then T can be realized as the full Hubbard tree of a quadratic polynomial $P(z) = z^2 + c$ with $P^b(0) = 0$ and with rotation number a/b at its α-fixed point. Thus T is dual to the lamination $\lambda \subset \Delta$ whose leaves join adjacent points in C and their preimages (see Figure 6 for the case $a/b = 1/3$).

Now a point $t \in S^1 = \partial\Delta$ corresponds to an end of T iff the ray from 0 to t in the unit disk crosses infinitely many leaves of λ. Thus we have:

Proposition 9.3 *A point $t \in S^1$ corresponds to an end of the quadratic tree $f : (T, p) \to (T, p)$ with rotation number a/b and $\delta(f) = 0$ iff its forward orbit does not land on the finite cycle $C \subset S^1$ with rotation number a/b.*

Corollary 9.4 *Every periodic point for $p_2|S^1$ is realized by a periodic end of T, except the cycle corresponding to the α-fixed point.*

This example suggests that the points in S^1 that are not realized by ends of T are analogous to the *ending lamination* that plays a prominent role in the theory of Kleinian groups (see e.g. [Th], [Bon1], [Min]).

Example: Two ends with the same marking. The map $\phi : \epsilon(T) \to S^1$ need not be injective. A non-injective example can be obtained by modifying the cubic tree shown in Figure 8 so the rotation number at p is still zero, but the critical points c and d have infinite forward orbits as shown in Figure 9. (Then p is a fan point of T.) In this example there is a ray α_c that starts at p, passes through c, and turns to the extreme right at every branch point; and a similar ray α_d that passes through d and turns to the left. There are no other rays in the interval $(\alpha_d, \alpha_c) \subset \epsilon(T)$; and since ϕ is monotone and its image is dense, this implies $\phi(\alpha_c) = \phi(\alpha_d)$.

The rays α_c and α_d correspond to two different infinite paths through the tiling on Δ shown at the right in Figure 9, both converging to $z = 1$.

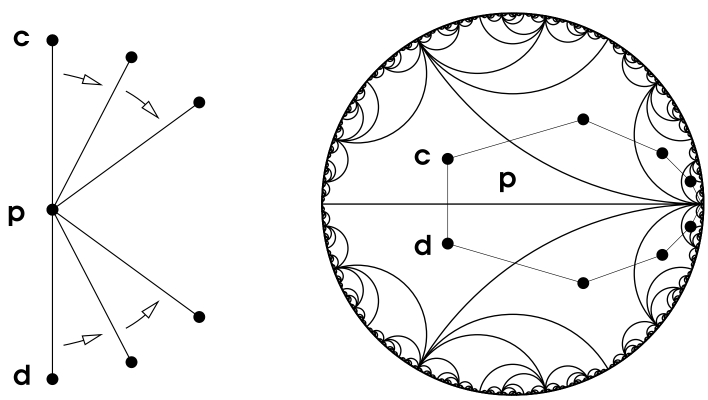


Figure 9. A cubic tree with two ends marked by the same point. The corresponding lamination has two tiled paths converging to $z = 1$.

Notes and references. For more on the Mandelbrot set and its limbs, see [DH] and [BF]. A detailed theory of quadratic laminations is presented in [Ke]; see also [Mc1, Ch. 6] and [Ki1].

10 Dynamics on the unit disk

Let $\Delta \subset \mathbb{C}$ denote the unit disk with the hyperbolic metric. In this section show that on a large scale, proper holomorphic maps $f : \Delta \to \Delta$ behave like branched coverings of trees. To label or *mark* the dynamics, we will also discuss the behavior of conjugacies between $f|S^1$ and $z \mapsto z^d$; and we will show that hyperbolic translation distances in Δ are tied to the values of $\log |f'|$ on the circle.

Throughout this section, d is fixed. The implicit and explicit constants in the results below are all allowed to depend on d.

The hyperbolic disk. Let $\Delta = \{z : |z| < 1\}$ be the unit disk, endowed with the hyperbolic metric $2|dz|/(1-|z|^2)$ of constant curvature -1. Let $d(a,b)$ denote the hyperbolic distance between a pair of points $a, b \in \Delta$, and let $[a,b] \subset \Delta$ denote the unique hyperbolic geodesic joining a to b. Note that we have

$$d(0,a) = |\log(1-|a|)| + O(1), \tag{10.1}$$

or equivalently $1 - |a| \asymp \exp(-d(0,a))$.

Blaschke products. Let $f : \Delta \to \Delta$ be a proper holomorphic map of degree $d > 1$. Let $C(f)$ denote the $(d-1)$ critical points of f. By the Schwarz lemma, f is a contraction: we have

$$d(f(a), f(b)) < d(a,b)$$

for any pair of distinct points $a, b \in \Delta$.

For $d \geq 2$ we let $\mathcal{B}_d$ denote the space of *Blaschke products* of the form

$$f(z) = z \prod_1^{d-1} \left(\frac{z - a_i}{1 - \overline{a}_i z} \right). \tag{10.2}$$

Note that $f(0) = 0$. Any proper holomorphic map of degree d with a fixed point in Δ is holomorphically conjugate to a map in $\mathcal{B}_d$.

Proposition 10.1 *The critical points of $f \in \mathcal{B}_d$ are no farther from the origin than its zeros. That is,* $\max_{c \in C(f)} d(0,c) \leq \max_1^{d-1} d(0,a_i)$.

Proof. Taking the logarithmic derivative, we find

$$z\frac{f'(z)}{f(z)} = 1 + \sum_1^{d-1} \frac{z(1-|a_i|^2)}{(z-a_i)(1-\overline{a}_i z)}. \tag{10.3}$$

Now for $|a| < |z| < 1$, we have

$$\operatorname{Re} \frac{(z-a)(1-\overline{a}z)}{z} = 1 + |a|^2 - \operatorname{Re}(az^{-1} - \overline{a}z) > 1 + |a|^2 - (1-|a|^2) = 0.$$

Since $\operatorname{Re}(1/w) = \operatorname{Re}(w)/|w|^2$, this implies

$$\operatorname{Re} \frac{z}{(z-a)(1-\overline{a}z)} > 0$$

as well. Thus if $\max |a_i| < |z| < 1$, equation (10.3) is a sum of terms with positive real parts, so $f'(z)$ cannot vanish. It follows that $|c| \le \max |a_i|$ for any critical point of f, and hence $d(0,c) \le \max d(0,a_i)$. ∎

Compactness and qualitative bounds. The normalization $f(0) = 0$ imposed on maps $f \in \mathcal{B}_d$ is especially useful because of the following compactness result.

Proposition 10.2 *Any sequence $f_n \in \mathcal{B}_d$ has a subsequence converging uniformly on compact sets to a proper map $h : (\Delta, 0) \to (\Delta, 0)$ with $1 \le \deg(h) \le d$.*

Proof. Pass to a subsequence so that the zeros of f_n (the a_i in formula (10.2)) converge to points in $\overline{\Delta}$, and that if $|a| \to 1$ then $(z-a)/(1-\overline{a}z) \to (-a)$ uniformly on compact subsets of Δ. ∎

This compactness easily implies several qualitative bounds for proper maps.

Corollary 10.3 *For each $r > 0$ there exists an $R > 0$ such that $f(B(0,R))$ contains the hyperbolic ball $B(0,r))$ for all $f \in \mathcal{B}_d$.*

Proof. If the statement fails, then we can find a sequence $f_n \in \mathcal{B}_d$, $R_n \to \infty$ and a point $y \in B(0,r)$, $r > 0$, such that $y \notin f_n(B(0,R_n))$. Passing to a subsequence, we obtain in the limit a proper map with $1 \le \deg(h) \le d$ whose image omits y, which is impossible. ∎

Corollary 10.4 *For any holomorphic map $f : \Delta \to \Delta$ of degree d and $x, y \in \Delta$, if $d(x,y) < r$ then $f^{-1}(y)$ is contained in an R-neighborhood of $f^{-1}(x)$.*

Proof. Normalize so that $f \in \mathcal{B}_d$ and $x = 0$, and apply the preceding result. ∎

Corollary 10.5 *If $f \in \mathcal{B}_d$ and $d(0, C(f))$ is large, then f is close to a rotation near 0.*

Proof. Suppose $f_n \in \mathcal{B}_d$ satisfy $d(0, C(f_n)) \to \infty$. Then any accumulation point h of f_n is a proper map with $h(0) = 0$ and no critical points. This implies $h(z) = e^{i\theta} z$ is a rotation, and hence f_n is close to a rotation when n is sufficiently large. ■

Corollary 10.6 *The 2-jet at z of any degree d holomorphic map $f : \Delta \to \Delta$ matches the 2-jet of a hyperbolic isometry with an error that tends to zero uniformly as $d(z, C(f)) \to \infty$.*

Proof. Normalize so that $f \in \mathcal{B}_d$ and $z = 0$, and apply the previous Corollary. ■

Corollary 10.7 *There is an $R > 0$ such that for any degree d holomorphic map $f : \Delta \to \Delta$ and any nontrivial geodesic segment $[a, b]$ with*

$$d([a, b], C(f)) > R,$$

we have $f(a) \neq f(b)$.

Proof. If we choose R large enough, then $f|[a, b]$ is so close to an isometry that the geodesic curvature of its image is nearly 0. But a path in hyperbolic space with geodesic curvature less than 1 (the curvature of a horocycle) cannot cross itself, so $f(a) \neq f(b)$. ■

Corollary 10.8 *For any $f \in \mathcal{B}_d$, the nearest zero controls the nearest critical point: we have*

$$d(0, C(f)) \leq \min\{d(0, a_i), 1 \leq i < d\} + O(1).$$

Proof. Since the endpoints of the segment $[0, a_i]$ are identified by f, there must be a critical point c within distance R of $[0, a_i]$. ■

Quantitative bounds. Using the last Corollary, we can make the previous bounds more quantitative.

Proposition 10.9 *For any $f \in \mathcal{B}_d$, we have*

$$\max(1 - |f'(0)|,\ |f''(0)|) = O(\exp(-d(0, C(f)))).$$

Proof. Clearly $A = f'(0) = \prod_1^{d-1}(-a_i)$ satisfies

$$1 - |f'(0)| = O(1 - \min |a_i|) = O(\exp(-\min d(0, a_i))),$$

and $\exp(-\min d(0, a_i)) = O(\exp(-d(0, C(f))))$ by the previous result. By similar reasoning we have

$$\sup_{|z|<1/2} |f(z) - Az| = O(1 - \min |a_i|),$$

which by Cauchy's integral formula gives $|f''(0)| = O(1 - \min |a_i|)$ as well. ■

Corollary 10.10 *The 2-jet at $z \in \Delta$ of a degree d holomorphic map $f : \Delta \to \Delta$ matches the 2-jet of a hyperbolic isometry with an error of $O(\exp(-d(z, C(f))))$.*

Proof. After composing with hyperbolic isometries, we can assume $z = f(z) = 0$ and $f \in \mathcal{B}_d$. Then the preceding result shows that the 2-jet of f at $z = 0$ is close to the 2-jet of a rotation of the disk. ■

Bounds along geodesics. We now use the fact that f converges to an isometry exponential fast away from $C(f)$ to control its behavior at large scales.

Theorem 10.11 *There is a constant $R > 0$ such that for any holomorphic map $f : \Delta \to \Delta$ of degree d:*

1. *If $d([a, b], C(f)) > R$, then $d(f(a), f(b)) = d(a, b) + O(1)$; and*
2. *If $d([a, b], f(C(f))) > R$, then $d(f^{-1}(a), f^{-1}(b)) = d(a, b) + O(1)$.*

Here f^{-1} is any branch of the inverse map which is continuous along $[a, b]$.

Proof. We begin with the proof of (1). Let $\alpha : [0, d(a,b)] \to [a,b]$ denote the parameterization of the geodesic from a to b by arclength. Then $\beta(s) = f(\alpha(s))$ parameterizes the image $f([a,b])$.

Suppose $d([a,b], C(f)) \geq R$. Let $\alpha(s_i)$, $i = 1, \ldots, d-1$ denote the projections of the critical points of f to $[a,b]$. Let $\kappa(s) \geq 0$ denote the curvature of $f([a,b])$ at $\beta(s)$, let $H(s) \subset \Delta$ denote the hyperplane normal to $f([a,b])$ at $\beta(s)$, and let

$$D(s) = \lim_{t \to 0+} \frac{d(H(s), H(s+t))}{t}.$$

Since the curvature controls the rate at which the normal plane advances with respect to arclength, we have

$$D(s) = |\beta'(s)|(1 + O(\kappa(s))).$$

Corollary 10.10 then implies

$$D(s) = 1 + O(\exp(-d(\beta(s), C(f)))).$$

By elementary hyperbolic geometry we also have

$$d(\alpha(s), C(f)) \geq \min_i |s - s_i| + d([a,b], C(f)) + O(1).$$

Finally, by choosing R large enough, we can assume $|\kappa(s)| \leq 1/2$ for all s and thus the hyperplanes $H(s)$ are disjoint. Combining these estimates, we find

$$\begin{aligned} d(f(a), f(b)) &\geq d(H(0), H(d(a,b))) \geq \int_0^{d(a,b)} D(s)\, ds \\ &= d(a,b) + O\left(\exp(-R) \int_0^{d(a,b)} \exp(-\min |s - s_i|)\, ds \right) \\ &= d(a,b) + O(\exp(-R)) = d(a,b) + O(1). \end{aligned}$$

The Schwarz lemma implies the reverse inequality, $d(a,b) \geq d(f(a), f(b))$, completing the proof of (1).

The proof of (2) is similar, using the fact that any branch of $f^{-1}(z)$ is exponentially close to an isometry when z is far from the critical values of f. ∎

Corollary 10.12 *Let $f : \Delta \to \Delta$ be a holomorphic map of degree d. Then any geodesic in Δ has a subdivision into $n \leq d$ consecutive segments,*

$$[a,b] = [c_0, c_1] \cup [c_1, c_2] \cup \cdots \cup [c_{n-1}, c_n],$$

such that $d(f(c_i), f(c_{i+1})) = d(c_i, c_{i+1}) + O(1)$ for all i, and $d(c_i, C(f)) = O(1)$ for $i = 1, \ldots, n-1$.

Proof. Take $c_1, \ldots, c_{n-1}$ to be the projections to $[a,b]$ of the critical points of f that lie within distance R of $[a,b]$, and apply the preceding Theorem. ■

Corollary 10.13 *For any $f \in \mathcal{B}_d$, the farthest critical point controls the farthest zero: we have*

$$\max_i d(0, a_i) \quad \leq \quad 2 \max_{c \in C(f)} d(0, c) + O(1).$$

Proof. Suppose $d(0, a_i)$ is maximized at $i = j$, and apply the preceding Corollary to the geodesic $[a,b] = [0, a_j]$. Then we have $d(c_1, C(f)) = O(1)$ and $d(0, c_1) \leq d(0, a_j)/2 + O(1)$, since

$$0 = d(f(0), f(a_j)) \geq d(0, c_1) - d(c_1, a_j) + O(1) = 2d(0, c_1) - d(0, a_j) + O(1).$$

■

Corollary 10.14 *We have* $\operatorname{diam} K \leq d \operatorname{diam}(f(K)) + O(1)$ *for any convex set $K \subset \Delta$.*

Proof. Apply Corollary 10.12 to a geodesic $[a,b] \subset K$ which nearly realizes its diameter, and consider the image of the largest subsegment $[c_i, c_{i+1}]$ of $[a,b]$. ■

Corollary 10.15 *Let $K, K' \subset \Delta$ denote the convex hulls of E and $E' = f^{-1}(E)$ respectively. Suppose E contains the critical values of f. Then $f^{-1}(K)$ is close to K': we have*

$$\sup_{x \,:\, f(x) \in K} d(x, K') = O(1).$$

Proof. Suppose $y = f(x)$ is in the convex hull K of E. Then y lies inside a triangle with vertices in E, and hence (since triangles are thin) within a bounded distance of a geodesic $[e_1, e_2]$ with endpoints in E. Since E includes the critical values of f, we can choose e_1, e_2 such that any critical value close to $[e_1, e_2]$ is close to e_1 or e_2. Then by Corollary 10.4 and Theorem 10.11, $f^{-1}[e_1, e_2]$ is close to a union of geodesics joining the preimages of e_1 and e_2, which necessarily lie in K'. Since x is close to the preimage of $[e_1, e_2]$, it is also close to K'. ∎

Markings. We conclude this section by relating the behavior of $f|\Delta$ to the behavior of $f|S^1$.

For any $f \in \mathcal{B}_d$, there is a unique *marking* homeomorphism

$$\phi : S^1 \to S^1$$

that varies continuously with f, conjugates f to $p_d(z) = z^d$, and satisfies $\phi(z) = z$ when $f = p_d$. This marking can be constructed using equation (2.3); see [Mc3]. We extend ϕ to a map $\overline{\phi} : \overline{\Delta} \to S^1$ by defining

$$\overline{\phi}(z) = \phi(z/|z|)$$

for $z \neq 0$ and (for convenience) $\overline{\phi}(0) = 1$.

Expansion and Hölder continuity. Let

$$M(f) = \sup_{z \in S^1} \log |f'(z)|$$

denote the *expansion factor* of $f \in \mathcal{B}_d$. Since $f|S^1$ has degree d, we have $M(f) \geq \log d$. Taking logarithmic derivatives and using equation (10.1), one readily verifies that

$$M(f) = \max_i d(0, a_i) + O(1), \tag{10.4}$$

where $0, a_1, \ldots, a_{d-1}$ are the zeros of f.

Theorem 10.16 *The expansion of f controls the Hölder exponent of its marking ϕ: for all $a, b \in S^1$ we have*

$$|\phi(a) - \phi(b)| = O(|a - b|^\beta),$$

where $\beta = (\log d)/M(f)$.

Proof. Let $J \subset S^1$ be the shortest arc connecting a and b, and let $\Lambda = \exp(M(f))$. Choose k such that

$$\Lambda^{-k-1} < |a-b| < \Lambda^{-k}.$$

Then $|f^k(a) - f^k(b)| < 1$ and so $f^k(J) \neq S^1$. But then $p_d^k(\phi(J)) \neq S^1$ as well, so the length of $\phi(J)$ is no more than $d^{-k} = (\Lambda^{-k})^\beta \leq d|a-b|^\beta$. ■

Corollary 10.17 *For any $z \in \Delta$ with $d(0,z) = r$, we have*

$$|\overline{\phi}(f(z)) - p_d(\overline{\phi}(z))| = O(e^{-\beta r}).$$

Proof. Let $w = z/|z|$. The hypothesis $d(0,z) = r$ implies $|z-w| = 1-|z| = O(e^{-r})$. By the maximum principle, we have $|f'| \leq \exp(M(f))$ throughout the unit disk, and thus $|f(z) - f(w)| = O(e^{M(f)-r})$. When combined with the preceding result, this bound yields

$$|\overline{\phi}(f(z)) - p_d(\overline{\phi}(z))| = |\overline{\phi}(f(z)) - \overline{\phi}(f(w))| = O(e^{\beta(M(f)-r)}) = O(e^{-\beta r}),$$

since $\beta M(f) = \log d$ and the implied constant is allowed to depend on d. ■

By similar reasoning we obtain:

Corollary 10.18 *For any $z, w \in \Delta$ with $d(0,z) = r > d(z,w) = s$,we have*

$$|\overline{\phi}(z) - \overline{\phi}(w)| = O(e^{-\beta(r-s)}).$$

Derivatives and translation distances. Let $C \subset S^1$ be a periodic cycle for the map $p_d(z) = z^d$. The *length* of C with respect a map $f \in \mathcal{B}_d$ with marking ϕ is defined by

$$L(C,f) = \log |(f^q)'(z)|,$$

where $q = |C|$ and $\phi(z) \in C$. The *length function* $Lf : S^1 \to \mathbb{R}_+$ is defined by

$$Lf(z) = \log |f'(\phi^{-1}(z))|.$$

Since

$$L(C,f) = \sum_{z \in C} Lf(z),$$

the length function records the lengths of all cycles.

Our final result shows Lf can be computed with bounded error by measuring how much f translates $z \in \Delta$ towards the origin (in the hyperbolic metric).

Proposition 10.19 *There is a constant $R > 0$ such that whenever*

$$d(0, f(z)) > R + \max_{c \in C(f)} d(0, f(c)),$$

we have

$$Lf(\overline{\phi}(z)) = d(z, 0) - d(f(z), 0) + O(1).$$

Proof. Let $w = f(z)/|f(z)|$. When R is large enough, the hypothesis implies there is a Euclidean ball $B(w, s)$ containing $f(z)$ and with $B(w, 100s)$ disjoint from the critical values of f. Let B' be the component of $f^{-1}(B)$ containing z. By the distortion theorems for univalent functions (see e.g. [Ah, Thm. 5-3]), B' is itself nearly a round ball centered on $z/|z|$, and the map $f : B' \to B$ nearly a similarity, expanding distances by the factor $\lambda = |f'(z/|z|)|$. Thus we have

$$Lf(\overline{\phi}(z)) = \log \lambda = \log \frac{1 - |f(z)|}{1 - |z|} + O(1) = d(z, 0) - d(f(z), 0) + O(1),$$

by equation (10.1). ∎

11 Trees as geometric limits

In this section we show that branched coverings of ribbon $\mathbb{R}$-trees arise naturally as limits of branched coverings of the hyperbolic plane. The main result is:

Theorem 11.1 *Let $f_n \in \mathcal{B}_d$ be a divergent sequence of Blaschke products. Then after passing to a subsequence, there is a minimal, degree d branched covering such that $f_n : (\Delta, 0) \to (\Delta, 0)$ converges geometrically to $f : (T, p) \to (T, p)$.*

It is easy to see that the limit is unique once it exists. We will also see that geometric limits inherit natural markings.

Limit of polygons. Given a metric space $(A, d(x, y))$, we let rA denote the rescaled space $(A, r \cdot d(x, y))$.

Recall that the *Hausdorff distance* between a pair of sets A, B in a metric space is defined by

$$d_H(A, B) = \max(\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)).$$

The *Gromov-Hausdorff distance* $d_{GH}(A, B)$ between a pair of abstract metric spaces is defined as the infimum of $d_H(A, B)$ over all metrics on $A \cup B$ extending the given metrics on A and B [Gr], [GH].

The space (T, p) will be constructed as a Gromov-Hausdorff limit of polygons in the unit disk, using the following well-known fact (cf. [Be1, Thm. 3.4]):

Proposition 11.2 *Let $K_n \subset \Delta$ be a sequence of hyperbolic k-gons with diameters $r_n \to \infty$. Then there is a subsequence such that $r_n^{-1} K_n$ converges in the Gromov-Hausdorff metric to a finite $\mathbb{R}$-tree T. The endpoints of T are limits of the vertices of K_n.*

We remark that a finite tree T is uniquely determined by the distances between its endpoints. Thus a sequence of rescaled hyperbolic k-gons converges, in the Gromov-Hausdorff metric, as soon as the k^2 distances between its pairs of vertices converge.

Orientation. A triple of points $(x, y, z) \in T^3$ is *positively oriented* if its convex hull $H \subset T$ is a tripod, and the ordering (x, y, z) on the endpoints of H is compatible with the ribbon structure on T. Similarly, a triple $(x, y, z) \in \Delta^3$ is *positively oriented* if its convex hull $H \subset \Delta$ is a triangle, and the ordering (x, y, z) on its vertices gives H the same orientation as that induced from the complex plane.

Limits of Blaschke products. Let $f : (T, p) \to (T, p)$ be a minimal branched covering of degree $d \geq 2$, and let f_n be a sequence in $\mathcal{B}_d$. Let

$$W = \overline{C}(f) \cup \overline{p} \subset T$$

denote the union of the grand orbits of the critical points of f and the basepoint p.

We say f_n converges to f *geometrically* if there exists a sequence of maps

$$h_n : (W, p) \to (\Delta, 0)$$

and a sequence of characteristic scales $r_n \to \infty$ such that the following conditions hold.

1. *Rescaling:* We have

$$d(x, y) = \lim r_n^{-1} d(h_n(x), h_n(y)) \tag{11.1}$$

for all $x, y \in W$;

2. *Conjugacy:* For all $x \in W$ we have

$$r_n^{-1} d(h_n(f(x)), f_n(h_n(x))) \to 0 \tag{11.2}$$

as $n \to \infty$; and

3. *Orientation:* Given any triple $x, y, z \in W$ such that (x, y, z) is positively oriented in T, the points

$$(h_n(x), h_n(y), h_n(z))$$

are positively oriented in Δ for all $n \gg 0$.

Proof of Theorem 11.1. Let f_n be a divergent sequence in $\mathcal{B}_d$, and let

$$r_n = \max_{z \,:\, f_n(z)=0} d(0, z). \tag{11.3}$$

Then $r_n \to \infty$ (since otherwise the zeros of f_n would lie in a compact subset in Δ, and we would have a convergent subsequence). By Proposition 10.1, we have

$$C(f_n) \cup f^{-1}(0) \subset B(0, r_n). \tag{11.4}$$

Neglecting finitely many terms, we can assume $r_n \geq 1$.

We will begin by constructing the tree (T, p). Let

$$W_n^k = \bigcup_{i=-k}^{k} f_n^i(C(f_n) \cup \{0\}) \subset \Delta.$$

Clearly $|W_n^k| = O(d^k)$, and W_n^k contains the critical values of f for $k \geq 1$. Using the initial bound (11.4) and Corollary 10.4, we find

$$\operatorname{diam} W_n^k = O(r_n \cdot d^k).$$

Let $T_n^k \subset \Delta$ be the hyperbolic convex hull of W_n^k. Then we have

$$1 \leq \operatorname{diam} r_n^{-1} T_n^k = O(d^k).$$

By Proposition 11.2, for each fixed k, we can pass to a subsequence such that $r_n^{-1} T_n^k$ converges in the Gromov-Hausdorff topology to a finite $\mathbb{R}$-tree T^k. Since $T_n^k \subset T_n^{k+1}$, we have isometric inclusions $T^1 \subset T^2 \subset \cdots$. The direct limit of these finite trees yields an $\mathbb{R}$-tree

$$T = \bigcup T_k.$$

Let $p \in T$ denote the limit of the basepoint $0 \in \Delta$ (which belongs to every T_n^k), let $W^k = \lim W_n^k \subset T_k$, and let $W = \bigcup W^k$.

Since each map $f_n | T_n^k \to T_n^{k+1}$ is distance non-increasing, we can pass to a further subsequence and obtain a distance non-increasing map $f : T \to T$ such that $f|T^k = \lim f_n | T_n^k$ for each k.

By the definition of Gromov-Hausdorff limits, there exist maps

$$h_n^k : W^k \to W_n^k \subset T_n^k$$

satisfying the rescaling condition (11.1) for each k. The conjugacy condition (11.2) holds automatically, by the definition of f. (Note that many points of W_n^k may coalesce to a single point of W^k, so some choices may be required in the construction of h_n^k.)

The endpoints of T^k are limits of vertices of the polygon T_n^k, which are cyclically ordered. By matching endpoints to vertices, we obtain a cyclic ordering on the endpoints of T^k, and hence a ribbon structure on T^k. There are only finitely many such ribbon structures, so after diagonalizing we can assume they converge for each k. Since the nearest-point projection $\partial T_n^{k+1} \to \partial T_n^k$ is order-preserving (monotone increasing), the ribbon structures on T^{k+1} and T^k are compatible. Hence we obtain a ribbon structure on T. By construction it satisfies the orientation condition in the definition of geometric limit whenever $x, y, z \in W$ are endpoints of some T^k; this easily implies the orientation condition holds for general triples.

Finally we show f is a minimal branched cover. Let $S_n^k = f_n^{-1}(T_n^k)$. Since T_n^k is the convex hull of W_n^k, Corollary 10.15 implies that S_n^k is close to the convex hull of $f^{-1}(W_n^k) \subset W_n^{k+1}$. Thus S_n^k is close to T_n^{k+1}. Passing to the limit, we find S_n^k converges to a subtree $S^k = f^{-1}(T^k) \subset T^{k+1}$.

Consider any edge $[a, b] \subset S^k$ connecting adjacent points in W^k. Since W^k contains the limits of the critical points $C(f_n)$, any critical point of f_n that is close to $[h_n(a), h_n(b)] \subset \Delta$ is close to its one of its endpoints. Thus by Theorem 10.11, f_n is nearly an isometry on the segment $[h_n(a), h_n(b)]$. It follows that $f|[a, b]$ is exactly an isometry, and therefore $f : S^k \to T^k$ is simplicial.

The planar embeddings of S_n^k determine a ribbon structure on S^k which in the limit is compatible with ribbon structure induced from T^{k+1}. Since $f_n : \Delta \to \Delta$ is a branched covering, so is $f : S^k \to T^k$.

Thus $f : (T, p) \to (T, p)$ is a branched covering of ribbon $\mathbb{R}$-trees. It is straightforward to check that f has degree d and that $C(f) = \lim C(f_n)$; thus $W = \overline{C}(f) \cup \overline{p}$, and therefore f is minimal. ∎

Markings. We conclude this section by showing that geometric limits can be chosen to respect markings.

Let $f : (T, p) \to (T, p)$ be a minimal branched covering of degree $d > 1$. For simplicity, assume f is not the cone over z^d. Then as we saw in §6, T has infinitely many ends, and there exists at least one marking

$$\phi : \epsilon(T) \to S^1.$$

Suppose $f_n \in \mathcal{B}_d$ converges geometrically to f, and let ϕ_n be the canonical markings of f_n (see §10). Suppose there is a marking $\phi : \epsilon(T) \to S^1$ such that whenever $x_i \in W$ converges to $\alpha \in \epsilon(T)$, we have

$$\phi(\alpha) = \lim_{i\to\infty} \lim_{n\to\infty} \overline{\phi}_n(h_n(x_i)). \tag{11.5}$$

We then say the convergence *respects markings*, and (f, ϕ) is the *marked limit* of the sequence $f_n \in \mathcal{B}_d$.

Theorem 11.3 *If $f_n \to f$ geometrically, then there is a subsequence such that the convergence respects markings.*

Proof. By Corollary 4.29, every sufficiently long segment $[a, b] \subset T$ meets $\overline{C}(f) \subset W$. Thus there is a constant $R > 0$ such that any $x, y \in W$ are connected by sequence $x = x_0, x_1, \ldots, x_n = y$ with $x_i \in W$ and $d(x_i, x_{i+1}) < R$.

Since W is countable and S^1 is compact, we can pass to a subsequence such that

$$\Phi(x) = \lim_{n\to\infty} \overline{\phi}_n(h_n(x)) \in S^1$$

exists for all $x \in W$. We claim there is a constant $\lambda > 1$ such that for all $x, y \in W$ with $d(x, y) < R$, we have

$$|\Phi(x) - \Phi(y)| = O(\lambda^{-d(0,x)}). \tag{11.6}$$

To see this, let $\beta_n = (\log d)/M(f_n)$ and recall from (10.4) that

$$M(f_n) = \max\{d(0, z) \ : \ f_n(z) = 0\} + O(1).$$

Since every zero of f_n converges geometrically to point in $f^{-1}(p)$, the rescaling equation (11.1) implies $M(f_n) = O(r_n)$, and hence $\beta_n > A r_n^{-1}$ for some $A > 0$. Letting $s_n = d(h_n(x), 0)$ and $t_n = d(h_n(x), h_n(y))$, we then have

$$\beta_n(s_n - t_n) \geq A r_n^{-1}(s_n - t_n) \to A(d(x, 0) - d(x, y)) \geq A d(x, 0) - R.$$

Applying Corollary 10.18, this implies

$$|\Phi(x) - \Phi(y)| = \lim_{n\to\infty} |\overline{\phi}_n(x) - \overline{\phi}_n(y)| = O(\exp(-Ad(x,0))),$$

which gives (11.6) with $\lambda = \exp(A) > 1$.

Now (11.6) easily implies the stronger bound

$$|\Phi(x) - \Phi(y)| = O(\lambda^{-d(p,[x,y])}) \tag{11.7}$$

for any $x, y \in W$. To see this, first assume x is the closest point to p on $[x,y]$. Then we can find a sequence $x = x_0, x_1, \ldots, x_n = y$ with $x_i \in W$, $d(x_i, x_{i+1}) < R$ and $d(0, x_i) \geq d(0,x) + iR/2$. Summing the geometric sequence resulting from (11.6), we obtain (11.7). To handle the general case, let $z \in [x,y]$ be the closest point to p and apply (11.7) to the segments $[x,z]$ and $[z,y]$.

Equation (11.6) implies that $\Phi(x_i)$ is a Cauchy sequence whenever x_i converges to an end of T, and hence the limit defining $\phi(\alpha)$ in (11.5) exists for every $\alpha \in \epsilon(T)$.

The same type of reasoning, using Corollary 10.17, shows that

$$|\Phi(f(x)) - p_d(\Phi(x))| = O(\lambda^{-d(0,x)})$$

for all $x \in W$. Passing to the limit, we find

$$\phi \circ f_* = p_d \circ \phi.$$

In fact, ϕ is a strong semiconjugacy: given $x \in W$ sufficiently far from p, the set $\Phi(f^{-1}(x))$ consists of d nearly equally space points on S^1, because the same is true of $\overline{\phi}_n(f_n^{-1}(z))$ when $d(0,z) \gg r_n$.

Monotonicity of ϕ follows from the orientation-preserving condition on the maps h_n. Consider a triple of distinct ends in T with $\alpha_2 \in (\alpha_1, \alpha_3)$ with respect to the cyclic ordering on $\epsilon(T)$. Let $x_i \in \alpha_i \cap W$ be close approximations to these ends. Then the triples (p, x_1, x_2), (p, x_2, x_3) and (p, x_3, x_1) are all positively oriented in T, so the same is true for their images under h_n, $n \gg 0$, as well as their projections to S^1 by $z \mapsto z/|z|$. Since $\phi_n : S^1 \to S^1$ also preserves orientation, upon passage to the limit as $n \to \infty$ we find

$$\phi(\alpha_2) \in [\phi(\alpha_1), \phi(\alpha_3)],$$

and thus ϕ is monotone increasing.

These last two observations imply ϕ is a marking of f; and by construction, f_n converges geometrically to the marked limit (f, ϕ). ■

12 Limiting length functions

In this section we prove the existence of a limiting length function $L(z)$ for a divergent sequence of Blaschke products $f_n \in \mathcal{B}_d$, and relate it to the length function of a limiting tree. The main results are:

Theorem 12.1 *Given any divergent sequence $f_n \in \mathcal{B}_d$, there is function $L : S^1 \to [0,1]$ such that after passing to a subsequence, the normalized lengths functions satisfy*

$$\frac{Lf_n(z)}{M(f_n)} \to L(z)$$

for all $z \in S^1$. The limiting length function $L(z)$ has at most $(d-1)$ laps.

Theorem 12.2 *Suppose f_n also converges geometrically to a marked branched covering*

$$f : (T,p) \to (T,p)$$

with $\epsilon(T) \neq \emptyset$. Then $L(z) = \overline{L}f(z)$ for all but finitely many $z \in S^1$. Moreover, if $\overline{L}f(z)$ has $(d-1)$ complete laps, then we have

$$\underline{L}f(z) \leq L(z) \leq \overline{L}f(z) \tag{12.1}$$

for all $z \in S^1$.

Theorem 12.3 *If $\epsilon(T) = \emptyset$, then $L(z) = 0$ outside a set of $\leq (d-1)$ points on S^1.*

Combining these results, we obtain:

Corollary 12.4 *The limiting length function $L(z)$ is piecewise constant.*

We have seen that in the quadratic case, $\overline{L}f(z)$ always has at least one complete lap, so (12.1) holds for all $z \in S^1$ when $d = 2$.

Laps and limits. We begin the proofs with some general remarks. As in §6, we say a function

$$g : S^1 \to [0,1]$$

has *e laps* if there exists a cyclically ordered sequence of distinct points $(z_1, z_2, \ldots, z_{2e})$ in S^1 such that the values $b_i = g(z_i)$ satisfy

$$b_1 < b_2 > b_3 < b_4 > b_5 \cdots < b_{2e} > b_1. \tag{12.2}$$

If the sequence can be chosen so that g is continuous at each z_i, and the values $b_1, b_2, \ldots, b_{2e}$ are alternately 0 and 1, then we say g has e *complete laps.*

Proposition 12.5 *If $g_j : S^1 \to [0,1]$ has at most e_j laps for $j = 1, 2, \ldots, n$ then $g(z) = \max g_j(z)$ has at most $\sum e_j$ laps.*

Proof. Suppose g has $e = 1 + \sum e_i$ laps, as evidenced by values $b_i = g(z_i)$, $i = 1, 2, \ldots, 2e$, satisfying (12.2). Then there is some j such that $g_j(z_i) = g(z_i) = b_i$ for more than e_j different even indices i. But $g_j(z_i) \le g(z_i) = b_i$ for all the odd values of i, so g_j itself has more than e_j laps. ∎

Proposition 12.6 *Any sequence of functions $g_n : S^1 \to [0,1]$ with at most e laps has a subsequence converging pointwise. The limit $h : S^1 \to [0,1]$ has at most e laps as well.*

Proof. Passing to a subsequence, we can assume that the turning points of g_n converge to a set $E \subset S^1$ with $|E| \le 2e$. We can also assume that g_n converges pointwise on a countable dense set $A \subset S^1$. The limit $h : I \cap A \to [0,1]$ is then monotone on each component I of $S^1 - E$, so it extends to a piecewise monotone function on S^1 with a countable set of discontinuities B. By local monotonicity, $g_n(z)$ converges to $h(z)$ outside $E \cup B$, and a further subsequence converges pointwise on this countable set as well. Clearly the limit function so obtained can have no more laps than the functions g_n that approximate it. ∎

Proposition 12.7 *Suppose $g_n : S^1 \to \mathbb{R}$ converges pointwise to a function h whose points of continuity are dense. Then there is a dense set $E \subset S^1$ such that $g_n(z_n) \to h(z)$ whenever $z_n \to z \in E$.*

Proof. Given $\epsilon > 0$, let

$$\begin{aligned} U_\epsilon &= \{z_0 \in S^1 \ : \ \text{there is a neighborhood } V \text{ of } z_0 \text{ and an } N > 0 \\ &\qquad \text{such that } |g_n(z) - h(z_0)| < \epsilon \text{ for all } n \ge N \text{ and } z \in J\}. \end{aligned}$$

Suppose h is continuous at z_0. Consider any closed interval I with z_0 in its interior such that $|h(z) - h(z_0)| < \epsilon/4$ for all $z \in I$. Let

$$F_N = \{z \in I \ : \ |g_n(z) - h(z_0)| \le \epsilon/2 \ \text{ for all } n \ge N\}.$$

Since g_n is continuous, F_N is closed; and since $g_n(z) \to h(z)$, we have $\bigcup F_N = J$. Thus by the Baire category theorem, there exists an $N > 0$ such that the interior V_N of F_N is nonempty. For all $z, z' \in V_N$ and $n \geq N$ we have

$$|g_n(z) - h(z')| \leq |g_n(z) - h(z_0)| + |h(z') - h(z_0)| < \epsilon/4 + \epsilon/2 < \epsilon,$$

and thus $V_N \subset U_\epsilon$. Since $V_N \subset I$ and I was an arbitrarily closed interval around z_0, this shows $z_0 \in \overline{U_\epsilon}$. By hypothesis, the set of points of continuity z_0 of h is dense, and therefore $\overline{U_\epsilon} = S^1$. The Proposition then holds for the dense G_δ given by $E = \bigcap_1^\infty U_{1/m}$. ∎

Proof of Theorem 12.1. Let $f \in \mathcal{B}_d$ have coefficients $(a_1, \ldots, a_{d-1})$, and for $z \in S^1$ let

$$g(z) = \max\left(0, \log(1 - |a_i|^2) - 2\log|z - a_i| \ : \ i = 1, \ldots, d-1\right).$$

Then Proposition 12.5 implies $g(z)$ has at most $(d-1)$ laps. On the other hand, by logarithmic differentiation we have

$$|f'(z)| = 1 + \sum_1^{d-1} \frac{1 - |a_i|^2}{|z - a_i|^2},$$

and for any $x_i \geq 1$ we have

$$\log\left(1 + \sum_1^{d-1} x_i\right) = \max(\log x_i) + O(1); \tag{12.3}$$

thus $\log|f'(z)| = g(z) + O(1)$.

Now consider a sequence $f_n \in \mathcal{B}_d$ with $M(f_n) \to \infty$. Since the number of laps is preserved by composition with the marking, the previous paragraph shows we can write

$$\frac{Lf_n(z)}{M(f_n)} = g_n(z) + O\left(\frac{1}{M(f_n)}\right),$$

where $g_n : S^1 \to [0,1]$ is a function with at most $(d-1)$ laps. Proposition 12.6 then implies there is subsequence such that $g_n(z)$ converges pointwise to a function $L : S^1 \to [0,1]$, also with at most $(d-1)$ laps; and since $M(f_n) \to \infty$, the same is true for $Lf_n(z)/M(f_n)$. ∎

Remark: Tropicalization. The replacement of a sum by a maximum, as in (12.3), is a basic theme in tropical geometry; see e.g. [St, Ch. 9] for details.

From the disk to the tree. We now proceed to the proof of Theorem 12.2. The following result relates translation lengths in the disk to translation lengths in the limiting tree.

Proposition 12.8 *Suppose $f_n : (\Delta, 0) \to (\Delta, 0)$ converges geometrically, respecting markings, to the branched covering*

$$f : (T, p) \to (T, p)$$

with marking ϕ. Then for all $\alpha \in \epsilon(T)$, there is a sequence $z_n \in S^1$ such that

$$z_n \to z = \phi(\alpha) \quad \textit{and} \quad \frac{Lf_n(z_n)}{M(f_n)} \to \frac{Lf(\alpha)}{M(f)}.$$

Proof. Let ϕ_n be the marking of f_n, let $W = \overline{C}(f) \cup \overline{p}$, let $h_n : (W, p) \to (\Delta, 0)$ be the approximate conjugacy between f and f_n defining the geometric limit, with associated scale $r_n \to \infty$. Then we have

$$\begin{aligned} r_n^{-1} M(f_n) &= r_n^{-1} \max\{d(0, z) + O(1) \,:\, f_n(z) = 0\} \\ &\to \max\{d(p, x) \,:\, f(x) = p\} = M(f) \end{aligned}$$

as $n \to \infty$.

Fix a ray α representing an end of T. Given $\epsilon > 0$, we can by equation (11.5) choose a vertex $x \in W \cap \alpha$ such that $|\overline{\phi}_n(h_n(x)) - \phi(\alpha)| < \epsilon$ for all $n \gg 0$. We can also assume

$$d(f(x), p) > \max\{d(f(c), p) \,:\, c \in C(f)\} \tag{12.4}$$

and

$$\tau(x) = d(x, p) - d(f(x), p) = M(f) Lf(\alpha).$$

Let $w_n = h_n(x) \in \Delta$ and let $z_n = \overline{\phi}_n(w_n) \in S^1$. Equation (12.4) and Proposition 10.19 then imply

$$Lf_n(z_n) = d(w_n, 0) - d(f(w_n), 0) + O(1)$$

for all $n \gg 0$, and thus

$$\begin{aligned} \frac{Lf_n(z_n)}{M(f_n)} &= \frac{r_n^{-1}(d(w_n, 0) - d(f(w_n), 0) + O(1))}{M(f_n)} \\ &\to \frac{d(x, p) - d(f(x), p)}{M(f)} = \frac{Lf(\alpha)}{M(f)} \end{aligned}$$

as $n \to \infty$. Since $|z - z_n| = |\phi(\alpha) - \overline{\phi}_n(w_n)| < \epsilon$ for all n sufficiently large, we can diagonalize to obtain a sequence $z_n \to z$ with the same limiting property. ∎

Proof of Theorem 12.2. It is convenient to pass to a subsequence such that the graphs of the functions

$$g_n = Lf_n/M(f_n) : S^1 \to [0,1]$$

converge, in the Hausdorff topology, to a closed relation $R \subset S^1 \times [0,1]$. By assumption we have $g_n(z) \to L(z)$ pointwise as well. Since $L(z)$ has at most $(d-1)$ laps, its points of continuity are dense, and so by Proposition 12.7 there is a dense set $E \subset S^1$ such that

$$R(z) = \{t \ : \ (z,t) \in R\} = \{L(z)\}$$

for all $z \in E$.

Suppose $\epsilon(T) \neq \emptyset$. We may assume, by rescaling the metric, that $M(f) = 1$. Then by Proposition 12.8, the relation R also contains the pushforward

$$\{(\phi(\alpha), Lf(\alpha)) \ : \ \alpha \in \epsilon(T)\}$$

of the graph of Lf, whose closure contains the graph of $\overline{L}f(z)$. Thus $L(z) = \overline{L}f(z)$ for all z in the dense set E.

Now recall from Corollary 7.6 that $\overline{L}f(z)$ is piecewise constant. Given an interval $I \subset S^1$ on which $\overline{L}f(z)$ is constant, there can only be finitely $z \in I$ such that $L(z) \neq \overline{L}f(z)$; otherwise $L(z)$ would have infinitely many laps. Thus $L(z) = \overline{L}f(z)$ outside a finite set.

Finally suppose $\overline{L}f(z)$ has $(d-1)$ complete laps. Then there is a cyclically ordered set $(z_1, \ldots, z_{2d-2})$ of points of continuity of $\overline{L}f(z)$ such that the values $b_i = \overline{L}f(z_i)$ are alternately 0 and 1. We can also assume $L(z_i) = \overline{L}f(z_i)$.

Suppose there is a point z such that $L(z) \notin [\underline{L}f(z), \overline{L}f(z)]$. Since $\overline{L}f$ is locally constant at each z_i, we can assume $z \neq z_i$ for all i. For concreteness, suppose $L(z) < \underline{L}f(z)$. Then there is a small interval (z_-, z_+) containing z, with no z_i in $[z_-, z_+]$, such that

$$L(z_-) = \overline{L}f(z_-) > L(z) < \overline{L}f(z_+) = L(z_+).$$

Consider the interval $[z_j, z_{j+1}]$ containing $[z_-, z_+]$. Assume, for example, that $b_j = 0$ and $b_{j+1} = 1$. Then we have

$$b_j = 0 < L(z_-) > L(z) < b_{j+1} = 1.$$

Inserting the points (z_-, z) between (z_j, z_{j+1}), we find that $L(z)$ has at least d laps, contrary to Theorem 12.1. ■

Remark. The same argument shows that if $L(z)$ and $\overline{L}f(z)$ have at most e and $\overline{e}$ laps respectively, then there are at most $e - \overline{e}$ points where $L(z) \notin [\underline{L}f(z), \overline{L}f(z)]$.

Proof of Theorem 12.3. Suppose $\epsilon(T) = \emptyset$. Then, as we saw in §6, f is a cone over z^d. Let $C(f) = \{c\}$ and let $P = f^{-1}(p)$. Then $d(x,p) = d(f(x),p)$ for any endpoint x of T that does not belong to the finite set P. Moreover, these endpoints satisfy $d(x,p) > d(c,p) = d(f(c),p)$. Thus one can apply Proposition 10.19 as above to conclude that $L(z) = 0$ on a dense subset of S^1. The fact that L has at most $(d-1)$ laps then implies $L(z) \neq 0$ for at most $(d-1)$ values of z. ■

13 Algebraic limits and strong convergence

The space of Blaschke products of degree d has a natural algebraic compactification $\overline{\mathcal{B}}_d$. Each point $(F,S) \in \partial\mathcal{B}_d$ is specified by a proper map F of degree strictly less than d, and a divisor of sources $S = \sum m_i s_i$ supported on the unit circle.

The suspension construction of §5 associates a branched covering $f : (T,p) \to (T,p)$ to each such boundary point (F,S). We say $f_n \in \mathcal{B}_d$ converges *strongly* to (F,S) if its algebraic limit is (F,S) and its geometric limit is the suspension of (F,S).

The pair (F,S) has a natural *marking relation* $\Phi : S^1 \to S^1$, which blows the sources up to intervals and transports the dynamics of (F,S) to that of $p_d(z) = z^d$. The points that are about to escape from the influence of a given source s are labeled by the finitely union of intervals

$$\Phi^*(s) = \{z \in \Phi(s) \ : \ z^d \notin \Phi(F(s))\} \subset S^1.$$

The main result of this section describes the behavior of lengths under strong limits in terms of these escape intervals. In these two result, 'almost all' means with just finitely many exceptions.

Theorem 13.1 *If $f_n \to (F,S) \in \partial\mathcal{B}_d$ strongly, then for almost all $z \in S^1$ we have*

$$\frac{Lf_n(z)}{M(f_n)} \to L(z) = \begin{cases} 1 & \text{if } z \in \Phi^*(s) \text{ for some } s \in \operatorname{supp} S; \\ 0 & \text{otherwise.} \end{cases} \tag{13.1}$$

When $|\operatorname{supp} S| = (d-1)$, *equality holds for all* $z \notin \bigcup \partial\Phi^*(s)$.

Corollary 13.2 *If* $f_n \to (F, S)$ *strongly, and then we have*

$$L(C) = \lim_{n\to\infty} \frac{L(C, f_n)}{M(f_n)} = \sum_{s \in \operatorname{supp} S} |C \cap \Phi^*(s)| \in \mathbb{Z}$$

for almost all cycles C. *When* $|\operatorname{supp} S| = (d-1)$, *equality holds for all cycles.*

Proof. The first statement follows from the formula $L(C) = \sum_{z\in C} L(z)$; the second, from the fact that $\bigcup \partial\Phi^*(s)$ contains no periodic points. ■

The algebraic compactification. We begin by summarizing results from [Mc3].

Since a map $f \in \mathcal{B}_d$ is determined by its zeros $(0, a_1, \ldots, a_{d-1})$, $\mathcal{B}_d$ can be naturally identified with the symmetric product $\Delta^{(d-1)}$. Taking its closure in $\mathbb{C}^{(d-1)}$, we obtain the *algebraic compactification*

$$\overline{\mathcal{B}}_d = \overline{\Delta}^{(d-1)} = \bigsqcup_{e=1}^{d-1} \Delta^{(e-1)} \times (S^1)^{(d-e)}.$$

We will identify a point $(a_i) \in \overline{\mathcal{B}}_d$ with the pair (F, S) consisting of the map

$$F(z) = z \prod_{|a_i|<1} \left(\frac{z - a_i}{1 - \overline{a}_i z}\right) \cdot \prod_{|a_i|=1} (-a_i) \tag{13.2}$$

and the *divisor of sources*

$$S = \sum_{|a_i|=1} 1 \cdot a_i \in \operatorname{Div}(S^1).$$

Note that $\deg(F) + \deg(S) = d$. We refer to the distinct points $(s_1, \ldots, s_k)$ in $\operatorname{supp} S$ as the *sources* of (F, S), and write $S = \sum_1^k m_i s_i$.

A sequence $f_n \in \mathcal{B}_d$ converges *algebraically* to (F, S) if the corresponding parameters converge in $\overline{\mathcal{B}}_d \cong \overline{\Delta}^{(d-1)}$. Equivalently, $f_n \to (F, S)$ if $f_n|\Delta \to F|\Delta$ uniformly on compact sets, and the zeros of $f_n|\Delta$ which escape the disk converge, as a divisor, to S.

Covering relations. Each boundary point $g = (F, S)$ can be interpreted dynamically as a multivalued-function or correspondence, as follows.

First note that for any $f \in \mathcal{B}_d$, there is a canonical lift of $f|S^1 = \mathbb{R}/\mathbb{Z}$ to a diffeomorphism $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ that varies continuously with f, satisfies $f(t+1) = f(t) + d$, and satisfies $\widetilde{f}(t) = dt$ when $f(z) = p_d(z) = z^d$. (This canonical lift exists because $\mathcal{B}_d$ is simply-connected.)

Suppose $f_n \to (F, S) \in \partial\mathcal{B}_d$. Since $f_n|S^1$ is expanding, the mappings $\widetilde{f}_n^{-1} : \mathbb{R} \to \mathbb{R}$ are contracting — in particular, they are equicontinuous. In fact one can show the lifted maps $\widetilde{f}_n^{-1}$ converge uniformly to a monotone increasing function $\widetilde{g}^{-1} : \mathbb{R} \to \mathbb{R}$ that depends only on (F, S). We regard its inverse as a multivalued function or *relation* $\widetilde{g} : \mathbb{R} \to \mathbb{R}$. The relation $\widetilde{g}$ sends points to compact intervals; more precisely, if $t \in \mathbb{R}$ and $t \bmod 1 = s_i$ is a source of (F, S), then $\widetilde{g}(s_i)$ is an interval of length m_i; otherwise, it is a single point.

The map $\widetilde{g}$ descends to a relation $g : S^1 \to S^1$ satisfying $g(s_i) = S^1$ if s_i is source of (F, S), and $g(z) = F(z)$ otherwise. We regard g as the graph of the degree d *covering relation* (F, S), which blows each source s_i up to an interval and then wraps it m_i times around the circle. The multiplicities of the sources are not visible in g, but they are recorded by its lift $\widetilde{g}$ which gives a geometric meaning to (F, S).

The marking of (F, S). Recall that each $f \in \mathcal{B}_d$ determines a canonical marking homeomorphism $\phi : S^1 \to S^1$ satisfying $\phi(f(z)) = p_d(\phi(z))$. Similarly (F, S) determines a marking *relation* $\Phi : S^1 \to S^1$, whose lift to the universal cover is given by

$$\widetilde{\Phi}(t) = \lim_{n\to\infty} d^{-n}\widetilde{g}^n(t)$$

(see [Mc3, Thm 2.1]). The relation Φ, which satisfies $\Phi(g(z)) = p_d(\Phi(z))$, is the Hausdorff limit of the markings ϕ_n of $f_n \to (F, S)$.

Marking the suspension. Our next task is to describe in detail the relationship between (F, S) and its suspension $f : (T, p) \to (T, p)$, defined in §5.

Let $A(T) = \{x \in T \,:\, d(x, p) = 1\}$. By the definition of the suspension, we can identify $A(T)$ with the grand orbits of the sources of (F, S): that is, we can assume that

$$A(T) = \bigcup_i \bigcup_{n\in\mathbb{Z}} F^n(s_i) \subset S^1,$$

that $f|A(T) = F|A(T)$, and that the induced cyclic ordering on $A(T) \subset S^1$ is compatible with the ribbon structure on T.

As in §8 we let $J(x) \subset \epsilon(T)$ denote the shadow of $x \in T - \{p\}$, i.e. the interval of ends represented by rays based at p that pass through x.

Proposition 13.3 *There is a unique marking ϕ for f such that for all $a \in A(T)$ with $J(a) \neq \emptyset$, we have $\Phi(a) = \overline{\phi(J(a))}$.*

Proof. If $|A(T)| = 1$ then f is the cone over z^d, $\epsilon(T) = \emptyset$ and the desired result is vacuously true. So we may assume $|A(T)| > 1$ and $\epsilon(T) \neq \emptyset$. Then $\Phi(a)$ is always a proper subinterval of S^1. Since any two markings of f differ by a rotation, uniqueness of the desired ϕ is clear.

We proceed to existence. Consider any $a \in A(T)$. By construction, $C(f) = \operatorname{supp} S$. If a is not a source of (F, S), then $f_*|J(a)$ is a homeomorphism, and we have:

$$J(F(a)) = f_*(J(a)).$$

If, however, a is a source of multiplicity m, then there are preimages $p_1, \ldots, p_m$ of p attached to a in T. In this case $J(a)$ can be expressed as a union of disjoint intervals

$$J(a) = I_0 \cup J(p_1) \cup I_1 \cup \cdots J(p_m) \cup I_m;$$

the map f_* restricted to each of these subintervals is injective; and their images are given by

$$f_*(J(p_i)) = \epsilon(T) - J(f(a)) \quad \text{and} \quad f_*(I_i) = J(f(a)). \tag{13.3}$$

In other words, f_* sends $J(a)$ m times completely around $\epsilon(T)$, covering $J(f(a))$ a total of $m+1$ times (once for each I_i, $0 \leq i \leq m$). But this is exactly the same as the behavior of the covering relation described by (F, S). Consequently, if we restrict the lift $\widetilde{g}$ of (F, S) to the universal cover $\widetilde{A}(T) \subset \mathbb{R}$ (which is dense), then there are lifts of J and f_* such that

$$\widetilde{J}(\widetilde{g}(a)) = \widetilde{f}_*(\widetilde{J}(a))$$

for all $a \in \widetilde{A}(T)$. Using this distinguished lift of f_*, we obtain a distinguished marking ϕ for f whose lift to the universal cover is given by

$$\widetilde{\phi}(\alpha) = \lim d^{-n} \lfloor \widetilde{f}_*^n(\alpha) \rfloor.$$

Since $\widetilde{J}(x)$ is contained in a bounded neighborhood of x, this implies that for all $a \in \widetilde{A}(T)$ with $\widetilde{J}(a) \neq \emptyset$ we have

$$\widetilde{\Phi}(a) = \lim d^{-n} \widetilde{g}^n(a) = \lim d^{-n} \widetilde{J}(\widetilde{g}^n(a)) = \lim d^{-n} \lfloor \widetilde{f}_*^n(\widetilde{J}(a)) \rfloor = \overline{\widetilde{\phi}(J(a))},$$

where the limits are taken in the Hausdorff topology on closed subsets of $\mathbb{R}$. Passing to the quotient $S^1 = \mathbb{R}/\mathbb{Z}$, it follows that $\Phi(a) = \overline{\phi(J(a))}$ as well.

■

It is now easy to determine the length function of f.

Proposition 13.4 *If there is a $c \in C(f)$ such that $\alpha \in J(c)$ but $f_*(\alpha) \notin J(f(c))$, then $Lf(\alpha)/M(f) = 1$. Otherwise $Lf(\alpha)/M(f) = 0$.*

Proof. The reversed edges of T are exactly those of the form $[p_i, c]$, where $p_i \neq p$ is a preimage of p, and c is the critical point at the midpoint of $[p_i, p]$. Each such edge has unit length, and different preimages of p have disjoint shadows, so we have $Lf(\alpha) = M(f) = 2$ on $\bigcup J(p_i)$ and $Lf(\alpha) = 0$ elsewhere. Since each preimage of p is associated to a critical point, we have $\bigcup J(p_i) \subset \bigcup_{C(f)} J(c)$; and by (13.3), if $\alpha \in J(c)$ then either $\alpha \in J(p_i)$ for some p_i, or $f_*(\alpha) \in J(f(c))$. ∎

Proposition 13.3 implies:

Corollary 13.5 *Let f be the suspension of (F, S). Then $\overline{L}f(z)/M(f) = 1$ if $z \in \Phi^*(s)$ for some source s; otherwise $\overline{L}f(z)/M(f) = 0$.*

Quadratic examples, *reprise.* Suppose $(F, S) \in \partial\mathcal{B}_2$. Then $F(z) = -sz$ and $S = 1 \cdot s$ for some point $s = \exp(2\pi i t) \in S^1$. If t is irrational, then $\Phi(s) = K(t)$ in the notation of §7, while $\Phi(F(s))$ is a single point. If $t = a/b$ is rational, then $\Phi(s) = I_{b-1}(a/b)$, while $\Phi(F(s)) = I_0(a/b)$. In either case, the preceding Corollary predicts $\overline{L}f(z)/M(f) = 1$ exactly when $z \in K(t)$, as is consistent with Theorem 8.2.

Proof of Theorem 13.1 . Let $L(z)$ be defined by the right side of equation (13.1) By the preceding Corollary, $\overline{L}f(z)/M(f)$ agrees with $L(z)$ for almost all z; and since $f_n \to (F, S)$ strongly, $\overline{L}f(z)/M(f) = \lim Lf_n(z)/M(f_n)$ for almost all z by Theorem 12.2.

If (F, S) has $(d-1)$ simple cycles, then $\overline{L}f(z)$ has $(d-1)$ complete laps, so $\underline{L}f(z) \leq Lf(z) \leq \overline{L}f(z)$ by Theorem 12.2, and Proposition 13.4 implies that $\underline{L}f(z) = \overline{L}f(z)$ outside the finite set of points $\bigcup \partial\Phi^*(s)$ where $\overline{L}f(z)$ jumps from 0 to 1. ∎

14 Examples of strong convergence

In this section we give effective criteria for strong convergence.

For simplicity, we begin with the quadratic case. Let $a_n \in \Delta$ be a sequence converging to $s = \exp(2\pi i t) \in S^1$. Then the Blaschke products

$$f_n(z) = z \left(\frac{z - a_n}{1 - \overline{a}_n z} \right)$$

converge algebraically to (F, S), where $F(z) = -sz$ and $S = 1 \cdot s$.

Theorem 14.1 *The quadratic maps $f_n \to (F, S)$ strongly if either:*

1. *The unique source $s \in \operatorname{supp} S$ is not a root of unity; or*
2. *We have $|a_n - s|^{2+\epsilon} = O_\epsilon(1 - |a_n|)$ for every $\epsilon > 0$.*

Using this result and a continuity argument, we will show:

Theorem 14.2 *Every minimal quadratic branched covering of a ribbon $\mathbb{R}$-tree arises as the geometric limit of a sequence in $\mathcal{B}_2$.*

Theorem 14.1 generalizes to higher degree as follows.

Let $f_n \to (F, S)$ be an algebraically convergent sequence in $\overline{\mathcal{B}}_d$. Let $a_n = (a_n(1), \ldots, a_n(d-1)) \in \Delta^{d-1}$ be the coefficients of f_n, and let $b = (b(1), \ldots, b(d-1))$ be the coefficients of (F, S). By a suitable choice of ordering, we can assume $|b(i)| = 1$ for $i = 1, 2, \ldots, e$, $|b(i)| < 1$ otherwise, and $a_n(i) \to b(i)$ for each i.

We say the zeros of f_n *escape at the same rate* if for every $\epsilon > 0$ and $1 \le i, j \le e$ we have

$$(1 - |a_n(i)|)^{1+\epsilon} = O_\epsilon(1 - |a_n(j)|).$$

(E.g. this is immediate if $|a_n(i)| = |a_n(j)|$, or $e = 1$.)

We say (F, S) has *simple sources* if each $s \in \operatorname{supp} S$ has multiplicity one.

Theorem 14.3 *Suppose (F, S) has simple sources, $f_n \to (F, S)$ algebraically, and the zeros of f_n escape at the same rate. Then f_n approaches (F, S) strongly provided:*

1. *There are no periodic sources, and distinct sources have disjoint orbits under F; or*
2. *For all $\epsilon > 0$ and $1 \le i \le d-1$, we have*

$$|a_n(i) - b(i)|^{2+\epsilon} = O_\epsilon(1 - |a_n(1)|).$$

Note that in (2) we have required that the escape rate of $a_n(1)$ controls the rate of convergence of all the zeros $a_n(i)$, even those with finite limits.

Corollary 14.4 *Any $(F, S) \in \partial\mathcal{B}_d$ with simple sources arises as a strong limit of $f_n \in \mathcal{B}_d$.*

Remark. One can show the same result holds without the assumption of simple sources; however these boundary points must be approximated by $f_n \in \mathcal{B}_d$ with *critical points* of higher multiplicity, not zeros of higher multiplicity.

Proof of Theorem 14.1. To show strong convergence, it suffices to show that the orbits of the critical points of f_n shadow their orbits under the limiting map F.

Let $F_n(z) = -a_n z$, and let c_n be the unique critical point of f_n. We will make use of the following facts.

1. We have $F_n(z) \to F(z)$ uniformly on $\overline{\Delta}$; indeed, we have
$$\sup_{z \in \Delta} |F_n(z) - F(z)| = |a_n - s| \to 0.$$
2. The functions F_n and F are uniformly Lipschitz on $\overline{\Delta}$ in the Euclidean metric.
3. For any $z \in \Delta$, we have
$$|f_n(z) - F_n(z)| \leq \frac{2(1 - |a_n|)}{1 - |z|}$$
(as follows by a simple computation).
4. The critical point of f_n satisfies $|c_n - a_n| = O(\sqrt{1 - |a_n|})$, and
$$1 - |c_n| \asymp \sqrt{1 - |a_n|}.$$
This can be checked by a computation, but it also follows from Corollary 10.12: c_n is a bounded hyperbolic distance from the midpoint of the geodesic $[0, a_n]$, since the endpoints of this segment are identified by f_n.

Let $c_n(k) = f_n^k(c_n)$, and let $\delta_n = \sqrt{1 - |a_n|}$. We claim that for fixed $k \geq 0$, we have
$$|c_n(k) - F^k(s)| = O_k(\delta_n + |a_n - s|). \tag{14.1}$$
For $k = 0$, this is immediate by (4) above. Assuming the bound for k, we can deduce it for $k + 1$ by first observing that (3) implies
$$|c_n(k+1) - F_n(f_n^k(c_n))| = O_k(\delta_n)$$

(using the fact that $|f^k(c_n)|$ is a decreasing function of k), then observing that the Lipschitz property (2) together with (14.1) implies

$$|F_n(c_n(k)) - F_n(F^k(s))| = O_k(\delta_n + |a_n - s|),$$

and finally observing that (1) implies

$$|F_n(F^k(s)) - F^{k+1}(s)| = O(|a_n - s|).$$

(The general principle at work here is that if g is uniformly close to h, and h is Lipschitz, then g^k is uniformly close to h^k.)

By Theorem 10.11, we also have

$$d(0, c_n(k)) = d(0, c_n) + O_k(1). \tag{14.2}$$

Let $r_n = d(0, c_n)$. Using these scale factors, pass to a subsequence such that f_n converges geometrically to a quadratic branched covering $f : (T, p) \to (T, p)$. For a fixed $k > 0$, let $K_n \subset \Delta$ denote the convex hull of the points $z = 0$ and $c_n(0), c_n(1), \dots c_n(k)$. By (14.2), the vertices of K_n are nearly equidistant from $z = 0$.

Now assume the rotation number t of F is irrational. Then the vertices of K_n converge (by (14.1) towards the $k+1$ distinct points $s, F(s), \dots, F^k(s) \in S^1$. Consequently the rescaled convex hulls $r_n^{-1} K_n$ converge, as metric spaces, to the unit cone over $k + 1$ distinct points. In this limit, the ribbon structure and the dynamics of f on T are consistent with the action of F on the circle; thus f is simply the suspension of (F, S), and therefore $f_n \to (F, S)$ strongly, as desired.

Finally assume $t = a/b$ is rational, so $F(z)$ has order b; and assume

$$|a_n - s|^{2+\epsilon} = O_\epsilon(1 - |a_n|),$$

or equivalently $|a_n - s| = O_\epsilon(\delta_n^{1-\epsilon})$ for all $\epsilon > 0$. Then (14.1) gives

$$|c_n(0) - c_n(b)| = O(\delta_n^{1-\epsilon}).$$

Since $1 - |c_n(0)|$ and $1 - |c_n(b)|$ are both comparable to δ_n, this implies their separation in the hyperbolic metric satisfies

$$\limsup \frac{d(c_n(0), c_n(b))}{d(0, c_n(0))} \leq \epsilon,$$

and hence $r_n^{-1} d(c_n(0), c_n(b)) \to 0$. Passing to the geometric limit, we find the unique critical point of $f : (T, p) \to (T, p)$ has period b and rotation number a/b, so once again f is isomorphic to the cone over (F, S). ∎

Proof of Theorem 14.2. By Theorem 8.1, a quadratic tree is determined by its rotation number $\rho(f)$ and, when $\rho(f)$ is rational, its fan height $\delta(f) \in [-1,1]$. The suspensions of boundary points $(F,S) \in \partial\mathcal{B}_2$ give all of these trees except those with $\delta(f) \neq 0$. By Theorem 14.1, all these suspensions arise as geometric limits.

To complete the proof, given any rational $a/b \in S^1$ and $0 \neq \delta_0 \in [-1,1]$, we will construct a sequence $h_n \in \mathcal{B}_2$ that converges geometrically to a branched covering h with $\rho(h) = a/b$ and $\delta(h) = \delta_0$.

Let $s = \exp(2\pi i a/b)$ and let $(F,S) \in \partial\mathcal{B}_2$ with the corresponding boundary point, with $F(z) = -sz$ and $S = 1 \cdot s$. Let $f : (T,p) \to (T,p)$ be the suspension of (F,S); then $\rho(f) = a/b$ and $\delta(f) = 0$, so $\overline{L}f(z)/M(f) = 1$ on $K(a/b)$ and 0 elsewhere, by Theorem 8.2.

Suppose for concreteness that $\delta_0 > 0$. Choose any point z_0 in the interior of $K_+(a/b)$; then $\underline{L}f(z_0) = \overline{L}f(z_0) = 0$. By Theorem 14.1, the sequence

$$f_n(z) = z(z - a_n)/(1 - \overline{a}_n z), \quad a_n = (1 - 1/n)s$$

converges strongly to (F,S); so by Theorem 12.2, we have

$$Lf_n(z_0)/M(f_n) \to 0.$$

Now consider a sequence of irrationals $t_n \to a/b$ from above, and let $s_n = \exp(2\pi i t_n)$. Then $K(t_n) \to K(a/b) \cup K_+(a/b)$ in the Hausdorff topology. Thus by Theorem 14.1 and a diagonalization argument, we have another sequence $g_n \in \mathcal{B}_2$ converging to (F,S) such that

$$Lg_n(z_0)/M(g_n) \to 1.$$

The intermediate value theorem then implies there is a sequence $h_n \to (F,S)$ such that

$$Lh_n(z_0)/M(h_n) \to \delta_0.$$

Pass to a subsequence such that h_n also converges geometrically, to a quadratic branched covering h. Then the proof of Theorem 14.1 shows that $\rho(h) = a/b$, while Theorem 12.2 implies $\overline{L}h(z_0) = \delta_0$, and hence Theorem 8.2 implies $\delta(h) = \delta_0$.

By Theorem 8.1, these two invariants determine h up to isomorphism, so we have shown that all quadratic trees arise as geometric limits. ∎

Proof of Theorem 14.3. The generalization of Theorem 14.1 to higher degree follows the same lines as the quadratic case. The key points are the following.

Let $s_i = b(i), i = 1, \ldots, e$ be the sources of (F, S) (each of multiplicity one); let

$$F_n(z) = \prod_{i=1}^{e}(-a_n(i)) \prod_{e+1}^{d} \frac{z - a_n(i)}{1 - \overline{a}_n(i)z};$$

and let $c_n(1), \ldots, c_n(d-1)$ be the critical points of f_n, labeled so that $c_n(i) \to s_i$ for $1 \le i \le e$. Then the basic facts used in the proof of Theorem 14.1 generalize as follows.

1. We have $F_n(z) \to F(z)$ uniformly on $\overline{\Delta}$; indeed, we have

$$\sup_{z \in \Delta} |F_n(z) - F(z)| \le \sum_{i=1}^{d-1} |a_n(i) - b(i)|.$$

2. The functions F_n and F are uniformly Lipschitz on $\overline{\Delta}$ in the Euclidean metric.

3. For any $z \in \Delta$, we have

$$|f_n(z) - F_n(z)| \le \sum_{i=1}^{e} \frac{2(1 - |a_n|)}{1 - |z|}.$$

4. The critical points of f_n satisfy $|c_n(i) - a_n(i)| = O(\sqrt{1 - |a_n(i)|})$, and

$$1 - |c_n(i)| \asymp \sqrt{1 - |a_n(i)|}.$$

In (1) and (3) we have used the fact that $|\prod x_i - \prod y_i| \le \sum |x_i - y_i|$ for points in the unit disk; and in (4) we have used Corollary 10.12.

The assumption that the zeros of f_n escape at the same rate implies, by (4), that for $1 \le i, j \le e$, we have

$$d(0, c_n(i))/d(0, c_n(j)) \to 1$$

as $n \to \infty$. Thus the corresponding critical points in the limiting tree (T, p) are equidistant from p. Facts (1-4) also imply, as before, that for $1 \le i \le e$ we have

$$|f_n^k(c_n(i)) - F^k(s_i)| = O_k\left(\delta_n + \sum |a_n(i) - b(i)|\right),$$

where $\delta_n = (\sum_1^e 1 - |a_n(i)|)^{1/2}$.

It follows that the orbits of the escaping critical points shadow the orbits of the sources, with sufficient accuracy that the limiting branched covering $f : (T, p) \to (T, p)$ is the suspension of (F, S). ■

Renormalization. Additional insight into Theorem 14.1 is provided by work of Epstein and DeMarco on renormalized limits of quadratic rational maps [Ep], [D1]. To explain this connection, suppose a/b is rational and $f_n \in \mathcal{B}_2$ is a sequence satisfying

$$a_n = -f_n'(0) \to s = -\exp(2\pi i a/b).$$

Then f_n converges algebraically to (F, S), where $F(z) = -sz$ is a rotation of period b, and $S = 1 \cdot s$.

If $a_n \to s$ along a horocycle in Δ, then

$$\frac{(-a_n)^b - 1}{\sqrt{1 - |a_n|^2}} \to iT$$

for some $T \in \mathbb{R}$. Under this assumption, Epstein shows that suitable conjugates of the mappings f_n^b of degree 2^b converge to the degree two limit

$$G_T(z) = z - 1/z + T,$$

locally uniformly on $\mathbb{C}^*$ [Ep, Prop. 2], [D1, §5]. In the limit, the unit circle (which is totally invariant for f_n) is replaced by the extended real axis (which is totally invariant for G_T).

Passing to a subsequence, we can also assume that f_n converges geometrically to a quadratic branched covering $f : (T, p) \to (T, p)$ with $\rho(f) = a/b$. Since $f_n^q \to G_T$, the lengths of the cycles of f_n that become cycles of G_T remain bounded in the limit.

In the notation of §7, these bounded cycles are naturally marked by points in $K_\pm(a/b)$. Indeed, the Julia set $J(G_T)$ itself is naturally marked by the invariant Cantor set for the degree two expanding map

$$p_2^b : K_-(a/b) \cup K_+(a/b) \to I_{b-1}(a/b).$$

(This Cantor set parameterizes the external rays landing on the boundary of the immediate basin of $z = 0$ for $P(z) = z^2 + c(a/b)$.)

Since multipliers of the cycles in this Cantor set remain bounded, $\overline{L}f(z) = 0$ at infinitely many points in $K_\pm(a/b)$. By Theorem 8.2, this implies $\delta(f) = 0$, and hence f_n converges strongly to (F, S) as predicted by Theorem 14.1.

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